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Periodic orbits near a bifurcating slow manifold

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Abstract

This paper studies a class of $1\frac{1}{2}$ -degree-of-freedom Hamiltonian systems with a slowly varying phase that unfolds a Hamiltonian pitchfork bifurcation. The main result of the paper is that there exists an order of $\ln^2 \epsilon^{-1}$ -many periodic orbits that all stay within an $\mathcal{O}(\epsilon^{1/3})$ -distance from the union of the normally elliptic slow manifolds that occur as a result of the bifurcation. Here $\epsilon \ll 1$ measures the time scale separation. These periodic orbits are predominantly unstable. The proof is based on averaging of two blowup systems, allowing one to estimate the effect of the singularity, combined with results on asymptotics of the second Painleve equation. The stable orbits of smallest amplitude that are persistently obtained by these methods remain slightly further away from the slow manifold being distant by an order $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$.

Keywords: Slow-fast systems; Hamiltonian systems; separatrix crossing; normally elliptic slow manifolds

1. Introduction

This paper considers a class of slow-fast $1\frac{1}{2}$ -degrees-of-freedom (d.o.f.) Hamiltonian systems, that includes the following example:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x(-\sin u + 2x^2), \\ \dot{u} &= \epsilon,\end{aligned}\tag{1}$$

where $u \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ is a slowly varying phase. As the systems considered in this paper, the example (1) is symmetric with respect to the reflection

$$\mathcal{R}: (x, y, u) \mapsto (-x, -y, u),\tag{2}$$

and with respect to a time-reversible symmetry

$$\mathcal{T}_\tau : (x, y, u)(t) \mapsto (x, -y, \tau - u)(-t), \quad (3)$$

for

$$\tau = \pi.$$

The more general symmetry \mathcal{T}_τ for $0 < \tau < 2\pi$ will be used later on. Example (1) also possesses a slow manifold of normally elliptic critical points of $(1)_{\epsilon=0}$:

$$S = \left\{ (x, y, u) | y = 0, x = \begin{cases} 0 & u \notin [0, \tau], \\ \pm \kappa(u) & u \in (0, \tau) \end{cases}, u \neq 0, \tau \right\}, \quad (4)$$

where $\kappa^2(u) = \frac{1}{2} \sin u > 0$ for $u \in (0, \pi)$ and $\tau = \pi$. See also Fig. 1. The set S is not uniformly normally elliptic because of the pitchfork bifurcations of $(1)_{\epsilon=0}$ at $u = 0$ and $u = \tau$. The main aim of this paper is to investigate the existence and stability of periodic orbits that remain close to S and therefore pass close to the bifurcation points for $\epsilon \ll 1$.

Many problems in physics can be reduced to a 2-d.o.f. Hamiltonian system with one d.o.f. being fast relative to another slow d.o.f., see e.g. [2, 5, 15, 22]. Such systems can be further reduced to slow-fast $1\frac{1}{2}$ -d.o.f. systems considered in this paper by reduction of energy. Either time t of a slow-fast system is *fast* as in (1) so that the velocities of the fast variables are $\mathcal{O}(1)$ while velocities of the slow ones are $\mathcal{O}(\epsilon)$. The system is then said to be *fast*. Otherwise time ϵt is *slow* such that the velocities of the fast variables are $\mathcal{O}(\epsilon^{-1})$ while velocities of the slow ones are $\mathcal{O}(1)$. In this situation the system is said to be *slow*. The limit $\epsilon = 0$ of the fast system gives the *layer problem* while the limit $\epsilon = 0$ of the slow system gives the *reduced problem*. For (1) the layer problem is

$$x' = y, \quad (5)$$

$$y' = -x(-\sin u + 2x^2), \quad (6)$$

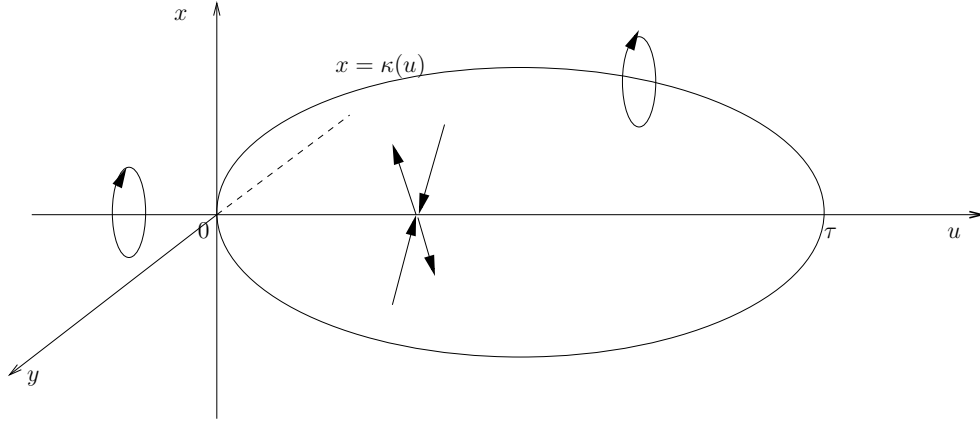
$$u' = 0.$$

The equations (5) and (6) for the fast variables x and y are called the *fast sub-system*. On the other hand, the reduced problem for (1) is

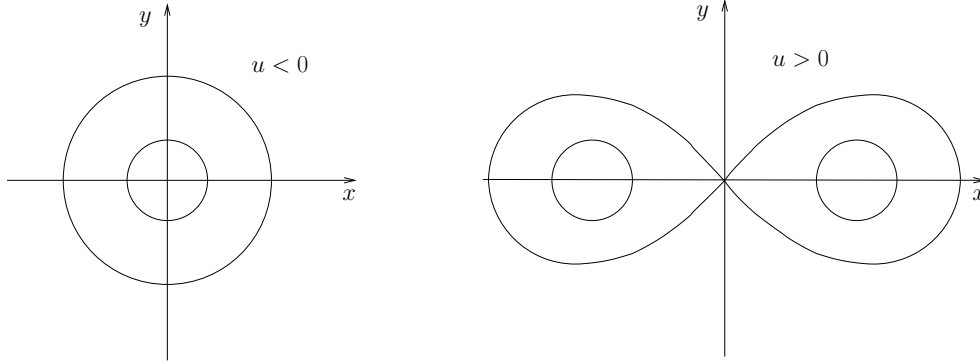
$$0 = y, \quad (7)$$

$$0 = -x(-\sin u + 2x^2),$$

$$u' = 1.$$



(a)



(b)

Figure 1: The slow manifold (a) and the dynamics of the frozen system (b). The variables x and y are fast whereas u is slow.

The reduced problem is only defined on critical points of the layer equations. The set of critical points make up the slow manifold for $\epsilon = 0$. In particular the set S is a set of elliptic critical points of the fast sub-system. In this paper, the resulting reduced system will be referred to as the *slow manifold approximation*. For (1) the slow manifold approximation is the system:

$$u' = 1, \quad (x, y, u) \in S, \quad u \in S^1, \quad (8)$$

obtained from (7) by restriction (x, y) to the normally elliptic slow manifold S . Orbits of this system reach the boundaries of S at $u = 0, \tau$ where S loses

normally ellipticity. But if we extend the system (8) to the closure of S then we obtain closed orbits

$$y_s = 0, |x_s(u)| = \begin{cases} 0 & u \notin (0, \tau), \\ \kappa(u) & u \in (0, \tau), \end{cases}, \quad (9)$$

which we will refer to as *singular closed orbits*. The periodic orbits considered in this paper remain close to the singular closed orbits in (9).

To describe the dynamics in 2-d.o.f. slow-fast Hamiltonian systems, one can often apply the theory of adiabatic invariants [1]. To explain this theory, first note that the layer problem, where the slow variables are fixed as parameters, is an integrable 1-d.o.f. system. Within a region of closed trajectories it is therefore possible to introduce action-angle variables, even in the full system for $\epsilon > 0$. Then, by averaging the Hamiltonian over the fast angle, one obtains a 1-d.o.f. system for the motion of the slow variables with the action appearing as a parameter. This is called the *adiabatic approximation*. Suppose that the trajectories within the phase plane of slow variables obtained from this approximation are closed. Then for analytic systems the theory says that, in general, the action only perpetually undergoes small oscillations $\mathcal{O}(\epsilon)$ [1, 6]. The phase space is, up to small gaps, filled with invariant tori [1] that are $\mathcal{O}(\epsilon)$ -close to the tori obtained from the adiabatic approximation.

A scenario, relevant to the problem considered here, where the theory of adiabatic invariants applies, is studied in [7]. Here the action-angle variables exist as a result of an elliptic equilibrium of the fast sub-system. Such an equilibrium varies smoothly by the implicit function theorem with respect to the slow variables to form a normally elliptic slow manifold [12]. The reference [7] then assumes analyticity and that the reduced problem gives rise to singular closed orbits within the normally elliptic slow manifold and show under these generic conditions that the slow manifold approximation accurately describes dynamics of the true system for ϵ sufficiently small. The references [19, 13, 10] also consider periodic solutions in a normally elliptic singular perturbation setting. The theory from [7] does not apply to (1) since the singular closed orbits (9) reach the boundaries of S where it loses normally ellipticity.

If there are symmetric separatrices on the phase plane of the fast variables described by the fast sub-system, such as in Fig. 1 (b), then the theory of adiabatic invariants needs some further modification, see [14, 17] for details. In [16, 25, 26] the authors presented very interesting results for a class of such

systems, also including the class of systems considered here in Eq. (10) below. They showed that in such systems there is in an order of ϵ^{-1} -many stable periodic orbits that repeatedly move from rotating within the separatrix lobes to rotating outside these lobes, see Fig. 1 (b). Moreover, there is an order 1 measure of regular motion; something that cannot be observed on Poincaré-sections: The resonance islands are small, being of order ϵ , and are therefore unlikely to be visible on Poincaré sections. A consequence of these results is that in systems with separatrix crossings there exist an order 1 set of initial conditions for which the action is perpetually invariant and the adiabatic approximation provides an accurate description of the full system. As opposed to the systems in [1] without separatrix crossings, there is also an order 1 set of chaotic dynamics where the adiabatic approximation does not give an accurate description of the dynamics. Crucial to the arguments in [16, 25, 26], however, is the condition that the crossings occur away from bifurcation points where the time scales are comparable. This condition was realized by taken the action to be greater than $c^{-1} > 0$, c large but independent of ϵ . The main result of this paper partially uncovers what changes when we move close to such bifurcation points and thus investigating the adequateness of the slow manifold approximation in systems with slow manifold bifurcations.

The geometric theory of singular perturbation provides another approach to the description of slow-fast systems. Although this theory is primarily applied to dissipative systems, focus being on normally hyperbolic slow manifolds, the view-point taken there is also relevant to mention in the context of this paper. This theory, also referred to as Fenichel's theory [3, 4], says that normally hyperbolic critical manifolds perturb to invariant slow manifold for ϵ sufficiently small. The invariant slow manifold is smoothly ϵ -close to the critical one. The flow restricted to the slow manifold converges to the flow of the reduced system, and the dynamics near the invariant slow manifold is, in some sense, inherited from the layer problem. Normally elliptic slow manifolds do not support such a general theory. However, the results of Gel-freich and Lerman in [7] show that, for general 2-d.o.f. analytic Hamiltonian slow-fast systems, the normally elliptic slow manifolds do in some sense also persist, potentially up to small gaps. The invariant slow manifold with gaps are filled with periodic orbits that are $\mathcal{O}(\epsilon)$ -close to the periodic orbits obtained from the slow manifold approximation. One of the aims of this paper is to land somewhere in-between these two different results and approaches, [16, 25] and [7], addressing periodic orbits in systems with separatrix cross-

ing, as [16, 25], while on the other hand relating this to normally elliptic slow manifolds and the slow manifold approximation as in [7]. Almost invariant normally elliptic slow manifolds have been studied in [11, 23].

In [21, 22], the authors studied different models of tethered satellites. One of these models is a finite-dimensional model that is obtained by replacing the tether connecting the satellite end-points with a spring that goes slack in compression. In [22], we showed for a Galerkin approximation of a more general PDE-model that such “slack spring” model, within this approximation, accurately describes the dynamics. In particular, it was shown that the motion remains close to the normally elliptic branches of a bifurcating slow manifold similar to the one shown in Fig. 1 (a) for a long period of time. The bifurcation of the slow manifold arose as a result of a pitchfork bifurcation within the limiting fast system; i.e. the situation also considering here in this paper. We did not explore a more quantitative description of the dynamics near these objects. The main result in this paper, however, applies to the Galerkin model in [22] (upon using the reduction of energy described in Section 2 below) and thus provide this example with a more detailed description of the dynamics in the vicinity of the bifurcating normally elliptic slow manifolds.

The treatment of bifurcating slow manifolds in Hamiltonian systems has also received attention elsewhere. For example, the references [5, 18, 20] considered the case of sub-critical pitchfork bifurcations in 2-d.o.f. Hamiltonian slow-fast systems. This was motivated by interfaces between ordered and disordered crystalline states. In particular, the references showed the persistence of singular heteroclinic solutions connecting equilibria on the normally hyperbolic critical manifold before and after the bifurcation. It was also shown that the heteroclinic connections remain close to the union of the normally hyperbolic branches of the slow manifolds before and after the perturbation. In particular, on the passage through the bifurcation the time scales were comparable. For this authors used Fenichel’s theory and the blowup method of Krupa and Szmolyan [9] to extend the normally hyperbolic slow manifolds near the bifurcation. The situation addressed here is related to these results in the sense that this paper studies the existence of periodic orbits that remain close to the union of the normally *elliptic* branches of the slow manifold.

Summary of main results. Before presenting the problem and stating the main results formally, the results of the paper are first described in words:

- (a) Close to the bifurcating normally elliptic slow manifold S there exist many unstable periodic orbits but, if any, fewer stable orbits. The instability of the periodic orbits is in contrast to the stability of S as stable critical points of the Hamiltonian fast sub-system.
- (b) The many unstable periodic orbits are $\mathcal{O}(\epsilon^{1/3})$ -close to the bifurcating normally elliptic slow manifold. For these orbits the passage through $u = 0$ and $u = \tau$ is described by a scaled system in which the time scales are comparable.
- (c) Stable solutions $\mathcal{O}(\epsilon^{1/3})$ -close to the bifurcating normally elliptic slow manifold can always be created (and destroyed) by varying the small parameter.
- (d) There are always stable periodic orbits further away from the bifurcating normally elliptic slow manifold at a distance of $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$. The larger distance from the slow manifold ($\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$ vs. $\mathcal{O}(\epsilon^{1/3})$ in (a), (b) and (c)) manifests itself in the fact that in the scaled system, used to describe the passage through $u = 0$ and $u = \tau$, the time scales are, in contrast to the orbits in (a), (b), and (c), not comparable: Close to the bifurcation points the fast variables undergo rapid oscillations.

Problem formulation and main result. I consider the following class of $1\frac{1}{2}$ -d.o.f. slow-fast Hamiltonian systems:

$$\begin{aligned}
H(u, v, x^2, y^2) &= v + \frac{1}{2}y^2(1 + M(x^2, y^2, u)) - \frac{1}{2}f(u)x^2 \\
&\quad + \frac{1}{2}x^4(1 + V(x^2, u)), \\
\omega &= dx \wedge dy + \epsilon^{-1}du \wedge dv,
\end{aligned} \tag{10}$$

with

$$M(0, 0, 0) = 0, \quad f(0) = 0, \quad V(0, 0) = 0, \quad \text{and} \quad f'(0) = 1, \tag{11}$$

which gives rise to the following fast system of equations:

$$\begin{aligned}
\dot{x} &= 2y\partial_{y^2}H, \\
\dot{y} &= -2x\partial_{x^2}H, \\
\dot{u} &= \epsilon.
\end{aligned} \tag{12}$$

I will motivate this choice further in Section 2 below. Here $\epsilon \ll 1$ and $u \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. The variable v is the conjugate to the slowly varying phase $u \in S^1$ and is introduced merely for later convenience. The functions M , f and V are assumed to be smooth and to satisfy the following conditions:

(A1) $f(0) = 0 = f(\tau)$ for some $\tau < 2\pi$;

(A2) $f(u) > 0$ for $u \in (0, \tau)$ and $f(u) < 0$ for $u \in (\tau, 2\pi)$;

(A3) f , $M(x^2, y^2, \cdot)$, $V(x^2, \cdot)$, and hence H , are symmetric about $u = \tau/2$:

$$f(u) = f(\tau - u), \quad M(x^2, y^2, u) = M(x^2, y^2, \tau - u), \quad V(x^2, u) = V(x^2, \tau - u);$$

(A4) The function M also satisfies $1 + M > 0$.

The assumption (A3) gives rise to the time-reversible symmetry \mathcal{T}_τ in (3): If $(x, y, u) = (x, y, u)(t)$ is a solution then so is

$$\mathcal{T}_\tau(x, y, u)(t) = (x, -y, \tau - u)(-t).$$

Given the form of M and V , an additional symmetry \mathcal{R} , see (2), has been enforced: If $(x, y, u) = (x, y, u)(t)$ is a solution then so is

$$\mathcal{R}(x, y, u)(t) \equiv (-x, -y, u)(t).$$

I will also think of $\mathcal{R} = -Id$ as the reflection acting on (x, y) alone.

The equilibrium $x = 0 = y$ of the fast sub-system, where u appears as a parameter, undergoes a symmetric pitchfork bifurcation at $u = 0$ and $u = \tau$. Cf. (11) $V = \mathcal{O}(u + x^2)$ and therefore by the implicit function theorem there exists two additional equilibria solving $\partial_{x^2} H = 0$ of the form $(x, y) = (\pm\kappa(u), 0)$ for $u > 0$ sufficiently small with

$$x^2 = \kappa(u)^2 = \frac{f(u)}{2} (1 + \mathcal{O}(u)), \quad (13)$$

solving

$$\partial_{x^2} H(u, v, x^2, 0) = -\frac{1}{2}f(u) + x^2 \left(1 + V + \frac{1}{2}x^2 \partial_{x^2} V \right) = 0. \quad (14)$$

Note also that $\kappa(\tau - u) = \kappa(u)$ by (A3). We require that the solution $x^2 = \kappa(u)^2$ exists and is positive for all $u \in (0, \tau)$, and ensure that $(x, y) = (\pm\kappa(u), 0)$ do not undergo additional bifurcations, by adding the following assumption:

(A5) The solution $x^2 = \kappa(u)^2$ of (14) is smooth, positive and satisfies

$$\partial_{x^2}^2 H(u, v, \kappa(u)^2, 0) > 0,$$

for all $u \in (0, \tau)$.

I shall, without loss of generality, henceforth take $\kappa(u) > 0$, $u \in (0, \tau)$. Note in (A5) that

$$\partial_x^2 H(u, v, \kappa(u)^2, 0) = 4\kappa(u)^2 \partial_{x^2}^2 H(u, v, \kappa(u)^2, 0),$$

since

$$\partial_{x^2} H(u, v, \kappa(u)^2, 0) = 0,$$

by construction. The condition (A5) therefore implies that $x = \pm\kappa(u)$, $y = 0$ are elliptic equilibria of the fast sub-system for every $u \in (0, \tau)$. I have illustrated the situation in Fig. 1 (a) and (b). Together with $x = 0$, $y = 0$ for $u \notin (0, \tau)$ these equilibria give rise to the slow manifold S . Following (A2) and (A5) the set $S \cup \{x = 0, y = 0\}$ is an isolated connected component of critical points for (12). Also $S \setminus \{u = 0, u = \tau\}$ is normally elliptic. Note, however, that S is not uniformly elliptic due to the bifurcations at $u = 0$ and $u = \tau$.

This paper investigates periodic orbits that remain close to S passing near the bifurcation points at $u = 0$ and $u = \tau$. To explain this differently: The reduced problem has a *singular closed orbit*, see (9). The periodic orbits of this paper lie close to the singular closed orbit in (9) in the sense that $||x(u)| - |x_s(u)|| + |y(u)|$ is small with respect to ϵ . There is another singular orbit $x = 0 = y$, $u \in S^1$ which enters the *normally hyperbolic* part: $x = 0$, $y = 0$, $u \in (0, \tau)$ of the slow manifold. Due to the \mathcal{R} -symmetry this is in fact a true orbit for all ϵ and it is cf. e.g. [26] typically highly unstable with multipliers of order $e^{\pm\mathcal{O}(\epsilon^{-1})}$.

When I later perform some numerical investigations I will base these on the example (1) where $f(u) = \sin u$, $M = 0 = V$, and $\tau = \pi$.

2. Reduction to (10) from a 2-d.o.f. system

One of the ways to obtain (10) from a more general setting, is to start from a natural slow-fast 2-d.o.f. system of the form

$$\begin{aligned} K &= \frac{1}{2}m_f(w, x^2)y^2 + \frac{1}{2}m_s(w)z^2 + W(w) + x^2Q(w, x^2), \\ \omega &= dx \wedge dy + \epsilon^{-1}dw \wedge dz, \end{aligned} \tag{15}$$

with $m_f, m_s > 0$, possessing a family of generic periodic orbits within the fix-point set $\{x = 0 = y\}$ of the symmetry action $(x, y) \mapsto \mathcal{R}(x, y) = (-x, -y)$, and reduce to a $1\frac{1}{2}$ -d.o.f. system by reduction of energy. Assume the following:

(C1) There exists a w_0 so that $Q(w_0, 0) = 0$, $\partial_w Q(w_0, 0) < 0$, $\partial_{x^2} Q(w_0, 0) > 0$;

(C2) The section $\{w = w_0\}$ is transverse to the family of periodic orbits.

It follows from (C1) that the fast sub-system:

$$\begin{aligned}\dot{x} &= m_f y, \\ \dot{y} &= -2x(Q + x^2 \partial_{x^2} Q + \frac{1}{2} \partial_{x^2} m_f y^2),\end{aligned}$$

with w fixed as a parameter, undergoes a super-critical pitchfork bifurcation at $w = w_0$ of $x = 0, y = 0$ and furthermore that there exists a solution $x^2 = \kappa(w)^2 > 0$ of $\partial_{x^2} Q(w, x^2) = 0$ within $w \in (w_0, w_1)$ for some $w_1 > w_0$ where $\partial_{x^2}^2 K(w, z, \kappa(w)^2, 0) > 0$. Consider then the reduced problem on $x^2 = \kappa(w)^2$:

$$\begin{aligned}K(w, z, \kappa(w)^2, 0) &= \frac{1}{2} m_s(w) z^2 + W(w) + \kappa(w)^2 Q(\kappa(w)^2, w), \\ \omega &= dw \wedge dz.\end{aligned}\tag{16}$$

We assume the following:

(C3) Initial conditions on the half-section $w = w_0, z > 0$ return to $w = w_0$ with $z < 0$ under the forward flow of (16).

This condition will allow us to show that the energy reduced system satisfies assumption (A5) above.

Within the region of closed orbits in $\{x = 0 = y\}$ we can replace (w, z) by action-angle variables $(\phi, I) \in S^1 \times \mathbb{R}$ and write

$$\begin{aligned}K(\phi, I, x^2, y^2) &= k(I) + \frac{1}{2} m_f(\bar{w}(\phi, I), x^2) y^2 + x^2 Q(\bar{w}(\phi, I), x^2), \\ \omega &= dx \wedge dy + \epsilon^{-1} d\phi \wedge dI.\end{aligned}\tag{17}$$

Here $k'(I) > 0$, $\phi = 0$ corresponds to $z = 0$ with $w > w_0$, and the orientation of ϕ is positive in the clockwise direction. Also $\bar{w}(\phi, I) = w$. It follows that

$$\partial_\phi \bar{w}(\phi, I) > 0,\tag{18}$$

for $\phi \in (0, \pi)$. Moreover, from (C3) it can be deduced that

$$\dot{\phi} > 0,$$

for the reduced problem (16). The invariance of K in (15) with respect to $(w, z, x, y) \mapsto (w, -z, x, -y)$ then becomes an invariance of K in (17) with respect to

$$(\phi, I, x, y) \mapsto (-\phi, I, x, -y). \quad (19)$$

In particular $\bar{w}(-\phi, I) = \bar{w}(\phi, I)$.

Proposition 1. *Suppose conditions (C1)-(C3). Then by reduction of energy, and appropriate scalings, the system (17) can be brought into (10) satisfying the conditions (A1)-(A5).*

PROOF. To reduce by energy we solve

$$K(\phi, I, x^2, y^2) = e, \quad (20)$$

for

$$I = \bar{I}(\phi, e, x^2, y^2). \quad (21)$$

The function \bar{I} is invariant with respect to $\phi \mapsto -\phi$ cf. (19). First, however, we shift the angle ϕ so that the origin is based at the bifurcation point $w = w_0, x = 0, y = 0$ where $I = k^{-1}(e)$ cf. (17) and (20). Let therefore $\tau/2 > 0$ be so that

$$\bar{w}(-\tau/2, k^{-1}(e)) = w_0, \quad x = 0, \quad y = 0, \quad (22)$$

and set

$$\phi = -\tau/2 + u.$$

Here $\tau = \tau(e) < 2\pi$. For simplicity, we henceforth continue to denote

$$\bar{w}(-\tau/2 + u, I), \quad K(-\tau/2 + u, I, x^2, y^2), \quad \text{and} \quad \bar{I}(-\tau/2 + u, e, x^2, y^2), \quad (23)$$

by

$$\bar{w}(u, I), \quad K(u, I, x^2, y^2), \quad \text{and} \quad \bar{I}(u, e, x^2, y^2), \quad (24)$$

respectively. The mapping $\phi \mapsto -\phi$ becomes

$$u \mapsto \tau - u, \quad (25)$$

and the resulting function $\bar{I}(u, e, x^2, y^2)$ is therefore invariant with respect to (25).

By Taylor expansion of (20) at $x = 0, y = 0$ we obtain the following local form of \bar{I} :

$$\begin{aligned} \bar{I}(u, e, x^2, y^2) = & k^{-1}(e) - (k^{-1})'(e) \left(\frac{1}{2} y^2 (m_f(w, 0) + \mathcal{O}(x^2 + y^2)) + Qx^2 \right. \\ & \left. + \left[\partial_{x^2} Q - (k^{-1})'(e) \partial_I \bar{w} \partial_w Q - \frac{(k^{-1})''(e)}{(k^{-1})'(e)} Q \right] x^4 + \mathcal{O}(x^6) \right). \end{aligned} \quad (26)$$

where Q and its partial derivatives are evaluated at $(w, x^2) = (\bar{w}(u, k^{-1}(e)), 0)$. By (22) we have that $\bar{w}(0, k^{-1}(e)) = w_0$, using (23) and (24), and by assumption (C1) it therefore follows that

$$Q(\bar{w}(0, k^{-1}(e)), 0) = 0, \quad \partial_w Q(\bar{w}(0, k^{-1}(e)), 0) < 0, \quad \text{and} \quad \partial_{x^2} Q(\bar{w}(0, k^{-1}(e)), 0) > 0.$$

Since $\dot{u} = \epsilon \partial_I K$ we obtain:

$$\begin{aligned} \frac{dx}{dt} &= \partial_y(-\bar{I}), \\ \frac{dy}{dt} &= -\partial_x(-\bar{I}), \\ \frac{du}{dt} &= \epsilon, \end{aligned} \quad (27)$$

by implicit differentiation of $K(u, \bar{I}(u, e, x^2, y^2), x^2, y^2) = e$ and introduction of a new time: $\epsilon^{-1}u$. The system (27) is a $1\frac{1}{2}$ -d.o.f. Hamiltonian system with Hamiltonian function

$$H = -\bar{I}(u, e, x^2, y^2). \quad (28)$$

By the invariance of \bar{I} with respect to (25), the system (28) possesses the time-reversible symmetry \mathcal{T}_τ in (3).

We now transform (28) into (10). To do this we first introduce $\tilde{\epsilon}$ by $\epsilon = (k^{-1})'(e)\tilde{\epsilon}$, rescale time $t = (k^{-1})'(e)\tilde{t}$ and introduce v conjugate to u so

that from (26) and (28)

$$\begin{aligned} H(u, v, x^2, y^2) &= v + \frac{1}{2}y^2(M_0 + \mathcal{O}(u + x^2 + y^2)) \\ &\quad - u(f_0 + \mathcal{O}(u))x^2 + x^4(V_0 + \mathcal{O}(u + x^2)), \\ \omega &= dx \wedge dy + \tilde{\epsilon}^{-1}du \wedge dv, \end{aligned}$$

where

$$M_0 = m_f(\bar{w}(0, k^{-1}(e), 0), \quad f_0 = -\partial_w Q(\bar{w}(0, k^{-1}(e)), 0)\partial_u \bar{w}(0, k^{-1}(e)),$$

and

$$V_0 = \partial_{x^2} Q(\bar{w}(0, k^{-1}(e)), 0).$$

These constants are positive by assumptions (C1), (C2) and (18). We henceforth drop the tildes. We then scale x , y , t and ϵ as follows:

$$x = \sqrt{\frac{f_0}{2V_0}}\tilde{x}, \quad y = \frac{f_0}{\sqrt{2V_0M_0}}\tilde{y}, \quad t = \frac{1}{\sqrt{f_0M_0}}\tilde{t}, \quad \epsilon = \sqrt{f_0M_0}\tilde{\epsilon},$$

and obtain the final Hamiltonian

$$\begin{aligned} H(u, v, x^2, y^2) &= v + \frac{1}{2}y^2(1 + \underbrace{\mathcal{O}(u + x^2 + y^2)}_{=M(u, x^2, y^2)}) - \underbrace{u(1 + \mathcal{O}(u))}_{=f(u)}x^2 \\ &\quad + \frac{1}{2}x^4 \left(1 + \underbrace{\mathcal{O}(u + x^2)}_{=V(u, x^2)} \right), \quad \omega = dx \wedge dy + \epsilon^{-1}du \wedge dv, \end{aligned}$$

upon again dropping the tildes, satisfying the conditions (A1)-(A4). Condition (A5) follows from (C3). This completes the proof. \square

The system in [22, Eq. (1.1.)] with $(u, U, v, V) = (w, z, x, y)$ and $w_0 = 1$ is of the form (15) and satisfies the conditions (C1), (C2) and (C3). The system can therefore be brought into (10) and hence the main result of this paper applies to this problem.

3. Main result

Following [7] we say that periodic orbits of (10) are *long* if they have periods of order $\mathcal{O}(\epsilon^{-1})$.

The main result of the paper is the following one:

Main result 1. *There exists an $\epsilon_0 > 0$ so that the following holds true for $\epsilon \leq \epsilon_0$:*

- 1° *There exists an order of $\ln^2 \epsilon^{-1}$ -many unstable, and long periodic orbits of (12) where $(x, y) = (x(u), y(u))$ remain $\mathcal{O}(\epsilon^{1/3})$ -close to the union of the normally elliptic critical manifold. Moreover, an order of $\ln^2 \epsilon^{-1}$ of these orbits are symmetric with respect to \mathcal{R} and/or \mathcal{T}_τ . The characteristic multipliers of the periodic orbits are $\mathcal{O}(\ln^{\pm 2} \epsilon^{-1})$.*
- 2° *For every $\epsilon \leq \epsilon_0$ there are fewer (typically with an order $o(\ln \epsilon^{-1})$), if any at all, stable periodic orbits of the type considered in 1° than unstable ones.*
- 3° *Take any $\epsilon_1 < \epsilon_0$ and consider $\epsilon \in I_1 = [\epsilon_1 - c_1 \epsilon_1^2, \epsilon_1 + c_1 \epsilon_1^2] \subset (0, \epsilon_0]$. Then within I_1 there will exist $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many closed intervals, each of length $\geq c_3^{-1} \epsilon_1^2 \ln^{-1} \epsilon_1$, of ϵ -values for which there exists at least one stable solution $(x, y) = (x(u), y(u))$ remaining $\mathcal{O}(\epsilon^{1/3})$ -close to the union of the normally elliptic critical manifold. Here c_1 , c_2 and c_3 may be large but they can be taken to be independent of ϵ_1 .*
- 4° *There exist stable, and long periodic orbits, symmetric with respect to \mathcal{R} and/or \mathcal{T}_τ , where $(x, y) = (x(u), y(u))$ remain $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$ -close to the union of the normally elliptic critical manifold.*

Remark 1. Regarding 1°: The periodic orbits are *long* since their periods are large being either $2\pi\epsilon^{-1}$ or $4\pi\epsilon^{-1}$. As explained in (b) above, the passage of these orbits through $u = 0$ and $u = \tau$ is described in a scaled system in which the time scales are comparable.

Regarding 2°: I have only been able to prove existence of stable orbits upon variation of ϵ (see 3°). In fact numerical computations on (1) seem to suggest that both existence and non-existence of stable periodic orbits of the type in 1° can occur.

Regarding 3°: The stable solutions within the intervals of 3° are distinct but the intervals could potentially overlap giving rise to several solutions for

fixed values of ϵ . Following 2° , however, the number of potential overlaps are typically $o(\ln \epsilon^{-1})$. Further details can be found in Section 6.6.

Regarding 4° : The orbits in 2° are rare, which we further demonstrate by performing some numerics on example (1), and the stable orbits in 4° are, in this sense, *typically* the smallest ones. These orbits are different from those in 1° in that they are stable but, moreover, their distance to the normally elliptic critical manifold is also larger: $O(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$ vs. $O(\epsilon^{1/3})$. In contrast to the orbits in 1° , this also means that the orbits undergo fast oscillations in the passage through the bifurcation points $u \in \{0, \tau\}$.

The minority of stable periodic orbits is in agreement with the results in [16, 25, 26] where $I \geq c^{-1} > 0$. These references show that there are asymptotically as many stable orbits as there are unstable ones. But, nevertheless, the unstable orbits are more frequent than the stable ones. See Table 1 in Section 6.4 for the results of numerical computations of periodic orbits for (1) and [26, Table 1] where they authors for an example of motion of charged particles in the Earth's magnetotail find 23865 unstable orbits and in comparison only 370 stable orbits. Furthermore, cf. Eq. (41) in [26], decreasing the action has the effect of diminishing the stability region. \diamond

Remark 2. Assumption (A3) could be relaxed: It is primarily included by convenience rather than necessity. The problems, such as the one in [22], that arise by the reduction in Section 2 do, however, satisfy this condition and I therefore found it natural to exploit this. \diamond

Remark 3. Each stable periodic orbit will in general give rise to stability islands. This was also the subject of interest in [25] showing that in the general case there are $O(\epsilon^{-1})$ -many stability islands of measuring at least $c^{-1}\epsilon$ for some c large but independent of ϵ . The islands due to the stable periodic orbits in 3° and 4° are expected to be smaller measuring $O(\epsilon \ln^{-3} \epsilon^{-1})$. I will also discuss this further in Section 7. \diamond

Outline. The main result is proved by obtaining fix points of a return map

$$\begin{aligned} P : \{(x, y, u) | u = -\pi + \tau/2\} &\rightarrow \{(x, y, u) | u = \pi + \tau/2\}, \\ (x, y, u = -\pi + \tau/2) &\mapsto P(x, y, u) = \phi_{2\pi/\epsilon}(x, y, -\pi + \tau/2), \end{aligned} \quad (29)$$

ϕ_t being the flow of (12). The return map is approximated using averaging and asymptotics of the second Painlevé equation. The averaging principle is

applied to two different blowup systems, one focusing in on $x = 0 = y$ for $u \in (-\pi + \tau/2, -u_*) \cup (\tau + u_*, \pi + \tau/2)$, with u_* small, and one focusing in on $x = \pm\kappa(u), y = 0$ for $u \in (u_*, \tau - u_*)$. Section 4 describes the blowup transformations used. These blowups are appropriate ϵ -dependent scalings. To describe the transition from $u = -u_*$ to $u = u_*$ I make use of the fact that (10), in a certain sense, is close to the second Painlevé equation, where there exists known asymptotics [8]. I present this asymptotics in Section 4 and show how it can be applied to (10). The small number u_* is written as $\mu^2 \hat{u}_*$ with μ small and connected to ϵ . The number \hat{u}_* is then fixed as $\mathcal{O}(1)$ with respect to ϵ . By connecting the μ with the blowups used in the fast space in the two separate regimes, I can obtain a lower bound ($\gg \epsilon$) of the normal frequency for $u \notin (-u_*, u_*) \cup (\tau - u_*, \tau + u_*)$. Such lower bound is crucial to successfully apply the averaging principle. The result, however, cannot be obtained by scalings alone; it is important to make use of the fact that the time-scale separation enhances as u moves away from 0 and τ to accurately approximate the return map. The averaging part is presented in Section 5 where I also describe the return map P in further details. Finally I solve the fix point equations and prove the main result in Section 6. Here I also perform some numerical investigations on the example (1).

Notation. To prove the result I make use of certain transformations of the fast variables for $u < -u_*$ and $u > u_*$. For this I will follow the following convention: Variables and functions relevant to the regime $u \in (-\pi + \tau/2, -u_*) \cup (\tau + u_*, \pi + \tau/2)$ are denoted by Roman letters, whereas Greek letters are used within $u \in (u_*, \tau - u_*)$. Scaled variables are given a hat: $\hat{\cdot}$. Some variables will be given subscripts starting from 0 to indicate that they later will be updated as a result of near-identity transformations. As always in work like this there will be introduced an abundance of constants. I will use c with and without subscripts for constants. They will always be independent of ϵ but I will often need them to be large. When I need a small constant I will therefore write it as c^{-1} . To avoid a long enumeration of constants, I will often restart an numeration at the beginning of a new section, a lemma, or even a new paragraph. It should be clear from the context where constants are related.

4. Blowup

In reference [24] the authors use the following blowup

$$x = \epsilon^{1/3}\check{x}, \quad y = \epsilon^{2/3}\check{y}, \quad u = \epsilon^{2/3}\check{u}, \quad (30)$$

to reduce (12) with $M = 0 = V$ to the Painleve equation of second kind:

$$\begin{aligned} \frac{d\check{x}}{d\check{u}} &= \check{y}, \\ \frac{d\check{y}}{d\check{u}} &= \check{u}\check{x} - 2\check{x}^3, \end{aligned} \quad (31)$$

ignoring here higher order terms that come from the expansion of $f = f(u)$ about $u = 0$. The reference presents asymptotics from [8] of (31) for $\pm\check{u}$ large that I will also make use of here. The asymptotics show that $\check{x} = \mathcal{O}(|\check{u}|^{-1/4})$, $\check{y} = \mathcal{O}(|\check{u}|^{1/4})$ for $\check{u} \rightarrow -\infty$. For $\check{u} \rightarrow \infty$, on the other hand, they show that $\check{x} \mp \sqrt{\hat{u}/2} = \mathcal{O}(\hat{u}^{-1/4})$ and $\check{y} = \mathcal{O}(\hat{u}^{1/4})$. This motivates the following blowup

$$\begin{aligned} \check{x} &= \delta^{1/4}\hat{x}, \\ \check{y} &= \delta^{-1/4}\hat{y}, \end{aligned} \quad (32)$$

when taking $\check{u} = \delta^{-1}\hat{u} < 0$. Then the asymptotics for $u < 0$ can be invoked by letting $\delta \rightarrow 0^+$. For $u > 0$ but small I will use an identical blowup of the deviation from $\pm\kappa(u) = \pm\sqrt{u/2} + \mathcal{O}(u^{3/2})$.

4.1. Blowup for $u \in [-(\pi - \tau/2), 0)$

Let $\mu = \epsilon^{1/3}\delta^{-1/2}$. Then motivated by the presentation above, in particular by (30) and (32), I introduce the following scaled variables \hat{x} , \hat{y} and \hat{u} :

$$\begin{aligned} x &= \epsilon^{1/3}\delta^{1/4}\hat{x} = \mu\delta^{3/4}\hat{x}, \\ y &= \epsilon^{2/3}\delta^{-1/4}\hat{y} = \mu^2\delta^{3/4}\hat{y}, \\ u &= \epsilon^{2/3}\delta^{-1}\hat{u} = \mu^2\hat{u}. \end{aligned} \quad (33)$$

It proves useful to write M and V as

$$\begin{aligned} M(x^2, y^2, u) &= uM_{00}(u) + x^2M_{10}(u) + y^2M_{01}(u) + \mathcal{O}((x^2 + y^2)^2), \\ V(x^2, u) &= uV_0(u) + \mathcal{O}(x^2), \end{aligned}$$

where the functions M_{00} , M_{10} , M_{01} , and V_0 are defined by Taylor's theorem applied to M and V at $(x^2, y^2) = 0$ and $x^2 = 0$, respectively. Inserting (33) into (10) then gives

$$\begin{aligned} H &= v + \frac{1}{2}\mu^4\delta^{3/2} \left(\hat{y}^2(1 + uM_{00}(u)) + \hat{F}(\hat{u})^2(1 + uM_{00}(u))^{-1}\hat{x}^2 \right) \\ &\quad + \frac{1}{2}\mu^4\delta^3\hat{x}^4 (1 + uV_0(u) + \mathcal{O}(\mu^2\delta^{3/2}\hat{x}^2)) + \frac{1}{2}\mu^4\delta^{3/2}\hat{y}^2 \left(\mu^2\delta^{3/2}\hat{x}^2 M_{10}(u) \right. \\ &\quad \left. + \mu^4\delta^{3/2}\hat{y}^2 M_{01}(u) + \mathcal{O}(\mu^4\delta^3\hat{x}^4 + \mu^8\delta^3\hat{y}^4 + \mu^6\delta^3\hat{x}^2\hat{y}^2) \right), \\ \omega &= \epsilon d\hat{x} \wedge d\hat{y} + \epsilon^{-1/3}\delta d\hat{u} \wedge dv. \end{aligned}$$

I have here introduced a scaled frequency \hat{F} defined by

$$\hat{F}(\hat{u})^2 = -\mu^{-2}f(\mu^2\hat{u})(1 + uM_{00}(u)) = -\hat{u} + \mathcal{O}(\mu^2). \quad (34)$$

At this stage I think of δ being small, allowing me to invoke the asymptotics of (31), but it can not be too small as we will need $\mu \ll 1$ and hence

$$\epsilon \ll \delta.$$

We will later need to quantify δ in relation to ϵ more accurately. In fact this will be one of the main difficulties.

Next, I introduce $v = \mu^4\delta^{3/2}\hat{v}$ and divide H by $\mu^4\delta^{3/2}$ to obtain a blowup Hamiltonian system $\hat{H} = \mu^{-4}\delta^{-3/2}H$:

$$\begin{aligned} \hat{H} &= h(\hat{u}, \hat{v}, \hat{x}, \hat{y}) + r(\hat{u}, \hat{x}, \hat{y}) + \mathcal{O}(\hat{y}^2(\mu^4\delta^3\hat{x}^4 + \mu^8\delta^3\hat{y}^6 + \mu^6\delta^3\hat{x}^2\hat{y}^2)), \quad (35) \\ \hat{\omega} &= \mu^{-4}\delta^{-3/2}\omega = \mu^{-1}d\hat{x} \wedge d\hat{y} + \mu^{-1}\delta^{-3/2}d\hat{u} \wedge d\hat{v}, \end{aligned}$$

introducing a leading order term:

$$h(\hat{u}, \hat{v}, \hat{x}, \hat{y}) = \hat{v} + \frac{1}{2}\hat{y}^2(1 + uM_{00}(u)) + \frac{\hat{F}(\hat{u})^2}{2}(1 + uM_{00}(u))^{-1}\hat{x}^2, \quad (36)$$

and a remainder:

$$r(\hat{u}, \hat{x}, \hat{y}) = \delta^{3/2} \left(\frac{1}{2}\hat{x}^4(1 + uV_0(u)) + \frac{1}{2}\hat{y}^2(\mu^2\hat{x}^2 M_{10}(u) + \mu^4\hat{y}^2 M_{01}(u)) \right).$$

Recall $u = \mu^2 \hat{u}$. The system (35) gives rise to the following equations of motions

$$\begin{aligned}\dot{\hat{x}} &= \mu \left(\hat{y}(1 + uM_{00}(u)) + \mathcal{O}(\delta^{3/2} \mu^2 \hat{y}(\hat{x}^2 + \hat{y}^2 \mu^2)) \right), \\ \dot{\hat{y}} &= -\mu \left(\hat{F}(\hat{u})^2 (1 + uM_0(u))^{-1} \hat{x} + 2\delta^{3/2} \hat{x}^3 (1 + uV_0(u)) + \mathcal{O}(\delta^3 \mu^4 \hat{x}(\hat{x}^2 + \mu^2 \hat{y}^2)) \right), \\ \dot{\hat{u}} &= \mu \delta^{3/2}.\end{aligned}$$

The truncation of the following representation of these equations

$$\begin{aligned}\frac{d\hat{x}}{d\hat{u}} &= \delta^{-3/2} \hat{y} + \mathcal{O}(\mu^2), \\ \frac{d\hat{y}}{d\hat{u}} &= \delta^{-3/2} (\hat{u} \hat{x} - 2\delta^{3/2} \hat{x}^3) + \mathcal{O}(\mu^2 \delta^{-3/2} \hat{x}),\end{aligned}\tag{37}$$

using here (33) and (34), coincide with the result of applying the scaling $\check{x} = \delta^{1/4} \hat{x}$, $\check{y} = \delta^{-1/4} \hat{y}$, $\check{u} = \delta^{-1} \hat{u}$ to the Painleve equations (31) considered in [24].

Action-angle variables. For $u < 0$ the function h in (36) is brought into action angle variables by introducing \hat{x}_0 and \hat{y}_0 through

$$\hat{x} = \hat{F}(\hat{u})^{-1/2} (1 + uM_{00}(u))^{1/2} \hat{x}_0, \quad \hat{y} = \hat{F}(\hat{u})^{1/2} (1 + uM_{00}(u))^{-1/2} \hat{y}_0.\tag{38}$$

Then

$$h_0 \left(\hat{u}, \hat{v}, \frac{1}{2} (\hat{x}_0^2 + \hat{y}_0^2) \right) \equiv h(\hat{u}, \hat{v}, \hat{x}, \hat{y}) = \hat{v} + \hat{F}(\hat{u}) \frac{1}{2} (\hat{y}_0^2 + \hat{x}_0^2).\tag{39}$$

This transformation is lifted to a symplectic transformation $(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \mapsto (\hat{x}_0, \hat{y}_0, \hat{u}_0, \hat{v}_0)$ on the full space via the generating function

$$G(\hat{x}, \hat{y}_0, \hat{u}, \hat{v}_0) = \delta^{-3/2} \hat{u} \hat{v}_0 + \hat{F}(\hat{u})^{1/2} (1 + uM_{00}(u))^{-1/2} \hat{x} \hat{y}_0,$$

and the equations

$$\begin{aligned}\hat{x}_0 &= \partial_{\hat{y}_0} G, \quad \hat{y} = \partial_{\hat{x}} G, \\ \hat{u}_0 &= \delta^{3/2} \partial_{\hat{v}_0} G = \hat{u}, \quad \hat{v} = \delta^{3/2} \partial_{\hat{u}} G = \hat{v}_0 - \frac{1}{2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)).\end{aligned}$$

Here I have in the last equality used the result from differentiating (34) with respect to \hat{u} . Applying this transformation to (35) gives

$$\hat{H} = h_0(\hat{u}, \hat{v}_0, \hat{z}_0) + r_0(\hat{u}, \hat{x}_0, \hat{y}_0) + \mathcal{O}(\hat{F}^{-5} \delta^3),\tag{40}$$

with h_0 the integrable part in (39) and a remainder

$$\begin{aligned}
r_0(\hat{u}, \hat{x}_0, \hat{y}_0) &= r(\hat{u}, \hat{x}, \hat{y}) + \hat{v} - \hat{v}_0 \\
&= \frac{1}{2} \delta^{3/2} \hat{F}(\hat{u})^{-2} \left(\hat{x}_0^4 (1 + u V_0(u)) \right. \\
&\quad + \hat{y}_0^2 f(u) (1 + u M_{00}(u)) (-M_{10}(u) \hat{x}_0^2 + f(u) (1 + u M_{00}(u)) M_{01}(u) \hat{y}_0^2) \\
&\quad \left. - \hat{x}_0 \hat{y}_0 (1 + \mathcal{O}(u)) \right), \tag{41}
\end{aligned}$$

and where the action-angle variables (\hat{z}_0, w_0) are the symplectic polar coordinates of (\hat{x}_0, \hat{y}_0) :

$$\hat{x}_0 = \sqrt{2\hat{z}_0} \cos w_0, \quad \hat{y}_0 = \sqrt{2\hat{z}_0} \sin w_0. \tag{42}$$

Remark 4. In (40) and (41) I have used (34) to eliminate μ and write these expressions only in terms of \hat{F} and δ . Note also how I in these expressions mix u and $\hat{u} = \mu^{-2}u$ together. The reason for introducing \hat{u} is that when $\hat{u} \leq -\hat{u}_* = \mathcal{O}(1)$ then this gives an order 1 lower bound of the scaled frequency:

$$\hat{F}(\hat{u}) \geq \hat{F}(\hat{u}_*),$$

provided μ is sufficiently small. The upper bound $\hat{F}(\hat{u})$ is of order $\mu(\epsilon)^{-1}$ and thus unbounded as $\epsilon \rightarrow 0$. Keeping track of how \hat{F} enters will be crucial when I am to decide what terms are important when I later wish to approximate the solution of these equations by means of averaging. On the other hand, I keep $M_{00}(u)$, $M_{10}(u)$, $M_{01}(u)$, and $V_0(u)$, for example, in terms of u to highlight that its estimate for u small is not particularly important. Instead it is important to highlight these terms are smooth and uniformly order 1 for all $u \leq 0$.

Big-Oh terms $\mathcal{O}(\hat{F}^{-q}\delta^p)$, like $\mathcal{O}(\hat{F}^{-5}\delta^3)$ in (40), are of order $\hat{F}^{-q}\delta^p$, $q, p > 0$ in the sense that such terms can be bounded *point-wise in \hat{u}* from above by $c\hat{F}(\hat{u})^{-q}\delta^p$, c independent of δ and \hat{u} , for all $\hat{u} \leq -\hat{u}_*$ and δ sufficiently small. I will return later to how different $\hat{F}^{-q}\delta^p$ can be compared.

Remark 5. It will later be shown that the action \hat{z}_0 only undergoes small oscillations. It will from this follow that

$$\begin{aligned}
x &= \mu \delta^{3/4} \hat{x} = \mu \delta^{3/4} (-\mu^{-2} f(u))^{-1/4} \sqrt{2\hat{z}_0} \cos w_0 \\
&= \epsilon^{1/2} (-f(u))^{-1/4} \sqrt{2\hat{z}_0} \cos w_0 = \mathcal{O}(\epsilon^{1/2})
\end{aligned}$$

using (34), (38) and (42) for $u \ll 0$. On the other hand when u is such that $\hat{F} = \mathcal{O}(1)$ then $x = \mathcal{O}(\mu\delta^{3/4}) = \mathcal{O}(\epsilon^{1/3}\delta^{1/4})$. \diamond

4.2. Blowup for $u \in (0, \tau/2]$

To present the asymptotics in [24] for the truncation of (37):

$$\begin{aligned} \frac{d\hat{x}}{d\hat{u}} &= \delta^{-3/2}\hat{y}, \\ \frac{d\hat{y}}{d\hat{u}} &= \delta^{-3/2}(\hat{u}\hat{x} - 2\delta^{3/2}\hat{x}^3), \end{aligned} \quad (43)$$

it is useful to introduce different blowup variables for $u \in (0, \tau/2)$ which I, as promised, will denote by Greek letters: $(\hat{\xi}, \hat{\sigma})$. They are obtained by performing the blowup (33) to the deviation $(\pm\xi, \pm\sigma)$ from $(x, y) = (\pm\kappa(u), 0)$:

$$\begin{aligned} (x, y) &= (\pm\kappa(u), 0) + (\pm\xi, \pm\sigma), \\ (\xi, \sigma) &= (\mu\delta^{3/4}\hat{\xi}, \mu^2\delta^{3/4}\hat{\sigma}). \end{aligned} \quad (44)$$

The particular form is based on the equivariance of the equations with respect to the action of \mathcal{R} . I also introduce symplectic polar coordinates $(\hat{\varrho}_0, \phi_0)$ by setting

$$\begin{aligned} \hat{\xi}_0 &\equiv \sqrt{2\hat{\varrho}_0} \cos \phi_0 = \hat{\Omega}^{1/2}(1 + \kappa(u)^2 M_0(u))^{-1/2} \hat{\xi}, \\ \hat{\sigma}_0 &\equiv \sqrt{2\hat{\varrho}_0} \sin \phi_0 = \hat{\Omega}^{-1/2}(1 + \kappa(u)^2 M_0(u))^{1/2} \hat{\sigma}, \end{aligned}$$

much as above. Here

$$\hat{\Omega}(\hat{u})^2 = 2\mu^{-2}\vartheta(u) = 2\hat{u}(1 + \mathcal{O}(u)), \quad (45)$$

is a scaled frequency with

$$\vartheta(u) = 2\kappa(u)^2 \partial_{x_2}^2 H(\kappa(u)^2, 0, u)(1 + uM^{00}(u))^{-1} = u + \mathcal{O}(u^2). \quad (46)$$

Also

$$uM^{00}(u) \equiv M(\kappa(u)^2, 0, u) = u \int_0^1 \frac{d}{ds} M(\kappa(su)^2, 0, su) ds.$$

Here $\vartheta(u) > 0$ for $u \in (0, \tau)$ by (A5). The blowup Hamiltonian, now denoted by $\hat{\Lambda}$, then becomes

$$\hat{\Lambda} = \zeta_0(\hat{u}, \hat{\nu}_0, \hat{\varrho}_0) + \rho_0(\hat{u}, \hat{\varrho}_0, \phi_0) + \mathcal{O}(\hat{\Omega}^{-7/2}\delta^{9/4} + \hat{\Omega}^{-5}\delta^3), \quad (47)$$

$$\hat{\omega} = \mu^{-1}d\hat{\xi}_0 \wedge d\hat{\sigma}_0 + \mu^{-1}\delta^{-3/2}d\hat{u} \wedge d\hat{\nu}_0. \quad (48)$$

splitting $\hat{\Lambda}$ it into an integrable part:

$$\zeta_0(\hat{u}, \hat{\nu}_0, \hat{\varrho}_0) = \hat{\nu}_0 + \hat{\Omega}_0(\hat{u})\hat{\varrho}_0,$$

and a remainder

$$\rho_0(\hat{u}, \hat{\varrho}_0, \phi_0) = \delta^{3/4}\hat{\Omega}(\hat{u})^{-1/2}\rho_{01}(\hat{u}, \hat{\varrho}_0, \phi_0) + \delta^{3/2}\hat{\Omega}(\hat{u})^{-2}\rho_{02}(\hat{u}, \hat{\varrho}_0, \phi_0),$$

where

$$\rho_{01}(\hat{u}, \hat{\xi}_0, \hat{\sigma}_0) = -\frac{1}{2}(1 + \mathcal{O}(u))\hat{\sigma}_0 + (1 + \mathcal{O}(u))\hat{\xi}_0^3 + uM^{10}(u)(1 + \mathcal{O}(u))\hat{\sigma}_0^2\hat{\xi}_0, \quad (49)$$

$$\begin{aligned} \rho_{02}(\hat{u}, \hat{\xi}_0, \hat{\sigma}_0) &= \frac{1}{2}(1 + \mathcal{O}(u))\hat{\sigma}_0\hat{\xi}_0 + \frac{1}{2}(1 + \mathcal{O}(u))\hat{\xi}_0^4 \\ &\quad + uM^{10}(u)(1 + \mathcal{O}(u))\hat{\sigma}_0^2\hat{\xi}_0^2 + 2u^2M^{01}(u)(1 + \mathcal{O}(u))\hat{\sigma}_0^4, \end{aligned} \quad (50)$$

with

$$\begin{aligned} M^{10}(u) &\equiv \partial_{x^2}M(\kappa(u)^2, 0, u), \\ M^{01}(u) &\equiv \partial_{y^2}M(\kappa(u)^2, 0, u). \end{aligned}$$

From (45) follows

$$\hat{\Omega}'(u) = \frac{1 + \mathcal{O}(u)}{\hat{\Omega}}, \quad (51)$$

$$\mu^2\hat{\Omega}^2 = 2u(1 + \mathcal{O}(u)), \quad (52)$$

which was used in (47) and will be used later on.

The details of the $\mathcal{O}(u)$ -terms in (47) are not important. It is again just important to highlight that they are smooth and uniformly bounded up until $u = 0$.

Remark 6. It will later be shown that the action $\hat{\varrho}_0$ only undergoes small oscillations. It will from this follow that $x \mp \kappa(u) = \mathcal{O}(\epsilon^{1/2})$ when $u \gg 0$. On the other hand when u is such that $\hat{\Omega} = \mathcal{O}(1)$ then $x \mp \kappa(u) = \mathcal{O}(\epsilon^{1/3}\delta^{1/4})$. \diamond

Having now introduced both (\hat{z}_0, w_0) and $(\hat{\varrho}_0, \phi_0)$, I am ready to present the asymptotics from [24] that I need.

Lemma 1. *Consider (43) and fix $c > 0$ large. Assume moreover that $\hat{z}_0 > 0$, $\hat{z}_0 \neq \ln 2/(2\pi)$ and that $\lambda = \lambda(\hat{z}_0, w_0)$ given by (61) belongs to the interval $[c^{-1}, \pi - c^{-1}] \cup [\pi + c^{-1}, 2\pi - c^{-1}]$. Then for $\hat{u} \geq \delta \hat{u}_*$ with \hat{u}_* sufficiently large the following asymptotics hold:*

$$\begin{aligned}\hat{x}_0(-\hat{u}) &= \sqrt{2\hat{z}_0} \cos w_0, \\ \hat{y}_0(-\hat{u}) &= \sqrt{2\hat{z}_0} \sin w_0,\end{aligned}\tag{53}$$

and

$$\begin{aligned}\hat{\xi}_0(\hat{u}) &= \sqrt{2\hat{\varrho}_0} \cos \phi_0, \\ \hat{\sigma}_0(\hat{u}) &= \sqrt{2\hat{\varrho}_0} \sin \phi_0,\end{aligned}\tag{54}$$

with the action-angle variables (\hat{z}_0, w_0) and $(\hat{\varrho}_0, \phi_0)$ related by the following expressions

$$w_0 = \frac{2}{3}\delta^{-3/2}\hat{u}^{3/2} + \frac{3}{2}\hat{z}_0 \ln(\delta^{-1}\hat{u}) - \frac{\pi}{2} + l,\tag{55}$$

$$\hat{\varrho}_0 = \frac{1}{2\pi} \ln \frac{1 + |p|^2}{2|\operatorname{Im} p|},\tag{56}$$

$$\phi_0 = -\frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}^{3/2} + 3\hat{\varrho}_0 \ln(\delta^{-1}\hat{u}) - \theta,\tag{57}$$

where

$$\theta(\hat{\varrho}_0, p) = Q(\hat{\varrho}_0) - \arg(1 + p^2),\tag{58}$$

$$Q(\hat{\varrho}_0) = -\frac{\pi}{4} + 7\hat{\varrho}_0 \ln 2 - \arg \Gamma(2i\hat{\varrho}_0),\tag{59}$$

$$p(\hat{z}_0, \lambda) = (e^{2\pi\hat{z}_0} - 1)^{1/2} e^{i\lambda},\tag{60}$$

$$\lambda = 3\hat{z}_0 \ln 2 - \frac{\pi}{4} - \arg \Gamma(i\hat{z}_0) - l.\tag{61}$$

If λ in (61) belongs to $[\pi + c^{-1}, 2\pi - c^{-1}]$ then the $+$ sign should be taken in (44). If $\lambda \in [c^{-1}, \pi - c^{-1}]$ then the $-$ sign should be taken (44).

Moreover, if $\hat{u} \geq \hat{u}_*$, with \hat{u}_* fixed, then the errors in (53) and (54) are of order $\delta^{3/2}$ and $\delta^{3/4}$, respectively.

Remark 7. The requirement $\hat{z}_0 \neq \ln 2/(2\pi)$ comes from the fact that $\hat{z}_0 = \ln 2/(2\pi)$ and $\lambda = \pi/2 \bmod \pi$ gives $1 + p^2 = 0$ where $\arg(1 + p^2)$, appearing in (58), is undefined. \diamond

PROOF. I use Eqs. (3)-(7) in [24] and write them in the blowup variables (\hat{x}_0, \hat{y}_0) respectively $(\hat{\xi}_0, \hat{\sigma}_0)$. Their s is my $\check{u} = \delta^{-1}\hat{u}$. Moreover, their α and ρ are related to my \hat{z}_0 and $\hat{\rho}_0$ by $\alpha^2 = 2\hat{z}_0$ and $\rho^2 = 2\hat{\rho}_0$. To relate their ϕ and θ to my w_0 and ϕ_0 I also end up having to solve the equations

$$\begin{aligned}\cos w_0 &= \sin \left(\frac{2}{3}\delta^{-3/2}\hat{u}^{3/2} + \frac{3}{2}\hat{z}_0 \ln(\delta^{-1}\hat{u}) + l \right), \\ \sin w_0 &= -\cos \left(\frac{2}{3}\delta^{-3/2}\hat{u}^{3/2} + \frac{3}{2}\hat{z}_0 \ln(\delta^{-1}\hat{u}) + l \right),\end{aligned}$$

and

$$\begin{aligned}\cos \phi_0 &= \cos \left(\frac{2\sqrt{2}}{3}\delta^{-3/2}(2\hat{u})^{3/2} - 3\hat{\rho}_0 \ln(\delta^{-1}\hat{u}) + \theta \right), \\ \sin \phi_0 &= -\sin \left(\frac{2\sqrt{2}}{3}\delta^{-3/2}(2\hat{u})^{3/2} - 3\hat{\rho}_0 \ln(\delta^{-1}\hat{u}) + \theta \right),\end{aligned}$$

with respect to w_0 and ϕ_0 . The solutions are

$$w_0 = \frac{2}{3}\delta^{-3/2}\hat{u}^{3/2} + \frac{3}{2}\hat{z}_0 \ln(\delta^{-1}\hat{u}) - \frac{\pi}{2} + l, \quad (62)$$

$$\phi_0 = -\frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u} + 3\hat{\rho}_0 \ln(\delta^{-1}\hat{u}) - \theta. \quad (63)$$

□

Remark 8. During the passing from $\hat{u} = -\hat{u}_*$ to $\hat{u} = \hat{u}_*$, as described in the previous lemma, the original variable $x = x(u)$ remains $\mathcal{O}(\epsilon^{1/3})$ -close to

$$\begin{cases} x = 0, & u < 0, \\ x = \pm\kappa(u), & u > 0. \end{cases} \quad \diamond$$

The assignment $(\hat{x}, \hat{y})(-\hat{u}) \mapsto (\hat{\xi}, \hat{\sigma})(\hat{u})$, $\hat{u} \geq \delta\check{u}_*$, in Lemma 1 is two-to-one due to its invariance with respect to the symmetry $(\hat{x}, \hat{y}) \mapsto \mathcal{R}(\hat{x}, \hat{y}) = (-\hat{x}, -\hat{y})$; in the action-angle variables (\hat{z}_0, w_0) the symmetry corresponds to a translation:

$$\mathcal{R} : (\hat{z}_0, w_0) \mapsto (\hat{z}_0, w_0 + \pi), \quad (64)$$

which we continue to denote by \mathcal{R} : The pair $(\hat{\phi}_0, \phi_0)$ in (56) and (57) is invariant with respect to \mathcal{R} . However, if I further assign the sign in (44) to the image of this assignment then I obtain a one-to-one mapping. Therefore consider $U = [c_1, c_2] \times S^1 \subset \{(\hat{z}_0, w_0)\}$ with $\ln 2/(2\pi) \notin [c_1, c_2]$ and remove a small, closed neighborhood of the singular set where $\lambda(\hat{z}_0, w_0) = 0$, π :

$$V = U \setminus \{(\hat{z}_0, w_0) \in U | \lambda(\hat{z}_0, w_0) \in [-c^{-1}, c^{-1}] \cup [-c^{-1} + \pi, \pi + c^{-1}]\}. \quad (65)$$

Note that V is open and large in measure, the complement having a measure of order c^{-1} with c large. Indeed, the set V is strip-like. The strips that are subtracted from V are closed sets having widths of order $c^{-1} \ln^{-1} \delta^{-1}$ cf. (55). The strips that are included in V , on the other hand, are open with non-empty interior having a width of order $\ln^{-1} \delta^{-1}$. There are $\ln \delta^{-1}$ of such strips. Then

Definition 1. for any $(\hat{z}_0, w_0) \in V$ I introduce P_{cr} and $P_{\text{cr}}^{\text{ext}}$ by $P_{\text{cr}}^{\text{ext}}(\hat{z}_0, w_0) = (\hat{\phi}_0, \phi_0, \eta)$ where the pair $(\hat{\phi}_0, \phi_0) = P_{\text{cr}}(\hat{z}_0, w_0)$ is the image of P_{cr} and given by (56) and (57). The argument $\eta \in \{\pm 1\}$ is $\eta = +1$ when $\lambda \in [\pi + c^{-1}, 2\pi - c^{-1}]$ and $\eta = -1$ when $\lambda \in [c^{-1}, \pi - c^{-1}]$. \square

The subscript cr in P_{cr} and $P_{\text{cr}}^{\text{ext}}$ is for “crossing.” The superscript ext is for “extended.” The assignment $P_{\text{cr}}^{\text{ext}} : V \rightarrow P_{\text{cr}}^{\text{ext}}(V)$ is a diffeomorphism as a flow map and for the truncated equations in (43) it is described by the asymptotics in Lemma 1. Moreover, it commutes with the translation $(\hat{z}_0, w_0) \mapsto \mathcal{R}(\hat{z}_0, w_0) = (\hat{z}_0, w_0 + \pi)$ in the following way:

$$P_{\text{cr}}(\mathcal{R}(\hat{z}_0, w_0)) = P_{\text{cr}}(\hat{z}_0, w_0), \quad \eta(\mathcal{R}(\hat{z}_0, w_0)) = -\eta(\hat{z}_0, w_0). \quad (66)$$

The fact that $P_{\text{cr}}^{\text{ext}}$ is still accurately described by the asymptotics in Lemma 1, when the remainder in (37) is included, is the subject of the following section.

4.3. Applying the asymptotics of the second Painleve Eq. to Eq. (37)

So far the asymptotics above is only valid for the truncation (43) of our blowup (37). The remainder in (37) that is ignored by the truncation is of order $\mu^2 \delta^{-3/2} |\hat{x}|$ to lowest order. To control the remainder I first need to estimate the truncation’s growth with respect to δ .

Lemma 2. *Suppose that the assumption of Lemma 1 holds true and consider $\hat{u} \in [-\hat{u}_*, \hat{u}_*]$. Then for δ sufficiently small there exists a constant $c > 0$ so*

that the assignment $\Psi_{\hat{u}} : (\hat{x}, \hat{y})(-\hat{u}_*) \mapsto (\hat{x}, \hat{y})(\hat{u})$ has the following growth properties with respect to δ :

$$|\hat{x}(\hat{u})| \leq c\delta^{-1/4}, |\hat{y}(\hat{u})| \leq c\delta^{1/4}, \quad 0 \leq \hat{u} < \check{u}_*\delta,$$

and

$$|\hat{x}(\hat{u})| \leq c\delta^{-3/4}\hat{u}^{1/2}, |\hat{y}(\hat{u})| \leq c\hat{u}^{1/4}, \quad \hat{u} \geq \check{u}_*\delta.$$

Here \check{u}_* is from Lemma 1.

Moreover, for $\hat{u} = \hat{u}_*$ the Jacobian of this assignment satisfies

$$|\partial\Psi_{\hat{u}_*}|, |\partial\Psi_{\hat{u}_*}^{-1}| \leq c\ln^2\delta^{-1}. \quad (67)$$

For any $\hat{u} \in (-\hat{u}_*, \hat{u}_*)$ the more pessimistic estimate applies:

$$|\partial\Psi_{\hat{u}}|, |\partial\Psi_{\hat{u}}^{-1}| \leq c\delta^{-1/4}\ln\delta^{-1}. \quad (68)$$

PROOF. The necessary estimation is delayed to Appendix A. \square

Using this lemma I can then describe how small ϵ should be relative to δ to be able to apply Lemma 1 to our blowup (37).

Lemma 3. *Lemma 1 applies to (37) provided*

$$\epsilon^{2/3}\delta^{-7/2}\ln^3\delta^{-1} \ll 1. \quad (69)$$

PROOF. Denote by $z = (\hat{x}, \hat{y})$ the solution of (37). Then I set

$$z(\hat{u}) = \Psi_{\hat{u}}(\Delta(\hat{u})), \quad (70)$$

with $\Delta(\hat{u})$ unknown and $\Psi_{\hat{u}}$ the flow-map from Lemma 2 associated with (43) with $\Psi_{-\hat{u}_*} = Id$. This last equality also implies that $z(-\hat{u}_*) = \Delta(-\hat{u}_*)$. By differentiating (70) with respect to \hat{u} I obtain an equation for $\Delta = \Delta(\hat{u})$:

$$\partial\Psi_{\hat{u}}\Delta'(\hat{u}) = \mathcal{O}(\mu^2\delta^{-9/4}).$$

I have here used Lemma 2 to conclude that for every \hat{u} the remainder of order $\mu^2\delta^{-3/2}|\hat{x}|$ is bounded by a term of order $\mu^2\delta^{-9/4}$. I then use (67) to invert $\partial\Psi_{\hat{u}}$ and integrate the resulting equation from $-\hat{u}_*$ to \hat{u}_* to obtain

$$\Delta(\hat{u}_*) = \Delta(-\hat{u}_*) + \mathcal{O}(\mu^2\delta^{-5/2}\ln\delta^{-1}) = z(-\hat{u}_*) + \mathcal{O}(\mu^2\delta^{-5/2}\ln\delta^{-1}).$$

Inserting this into (70) finally gives:

$$z(\hat{u}_*) = \Psi_{\hat{u}_*}(z(-\hat{u}_*)) + \mathcal{O}(\mu^2\delta^{-5/2}\ln^3\delta^{-1}),$$

where we have also used (68) to expand $\Psi_{\hat{u}_*}(\Delta(\hat{u}_*))$. Inserting $\mu^2 = \epsilon^{2/3}\delta^{-1}$ completes the proof. \square

5. The return map

It is natural to view the return map P in (29) as a stroboscopic mapping that assigns initial conditions (\hat{z}_0, w_0) at $u = -\pi + \tau/2$ to final conditions 2π -later at $u = \pi + \tau/2$. That is

$$(\hat{z}_0, w_0) \mapsto P(\hat{z}_0, w_0) = \phi_{2\pi}(\hat{z}_0, w_0; -\pi + \tau/2),$$

with $\phi_u(\hat{z}_0, w_0; u_0)$, satisfying $\phi_{u_0}(\hat{z}_0, w_0; u_0) = (\hat{z}_0, w_0)$, being the flow of the non-autonomous system (37). For simplicity I will write this as

$$(\hat{z}_0, w_0)(u = -\pi + \tau/2) \mapsto P(\hat{z}_0, w_0) = (\hat{z}_0, w_0)(u = \pi + \tau/2). \quad (71)$$

I will decompose the mapping P into the following two parts

$$P_1 : (\hat{z}_0, w_0)(u = -\pi + \tau/2) \mapsto (\hat{\rho}_0(u = \tau/2), \phi_0(u = \tau/2), \eta),$$

and

$$P_2 : (\hat{\rho}_0(u = \tau/2), \phi_0(u = \tau/2), \eta) \mapsto (\hat{z}_0, w_0)(u = \pi + \tau/2),$$

so that $P = P_2 \circ P_1$. Here I have adopted a similar notation to the one used in (71). I further decompose P_1 into three parts setting

$$P_1 = P_i \circ P_{\text{cr}}^{\text{ext}} \circ P_o,$$

where $P_{\text{cr}}^{\text{ext}}$ is as in Definition 1, which following Lemma 3 is accurately described by the asymptotics in Lemma 1, and where

- P_o is the “outer” map

$$P_o : (\hat{z}_0, w_0)(u = -\pi + \tau/2) \mapsto (\hat{z}_0, w_0)(\hat{u} = -\hat{u}_*); \quad (72)$$

- P_i is the “inner” map:

$$P_i : (\hat{\rho}_0, \phi_0)(\hat{u} = \hat{u}_*) \mapsto (\hat{\rho}_0, \phi_0)(u = \tau/2). \quad (73)$$

In principle, to make sense of $P_i \circ P_{\text{cr}}^{\text{ext}}$, I should include η in the argument of P_i , but by the \mathcal{R} -symmetry P_i applies as two identical copies on the components $\eta = \pm 1$. For ease of notation I just think of P_i acting on the P_{cr} -part only (recall the definition of P_{cr} in Definition 1). The maps are illustrated in Fig. 2.

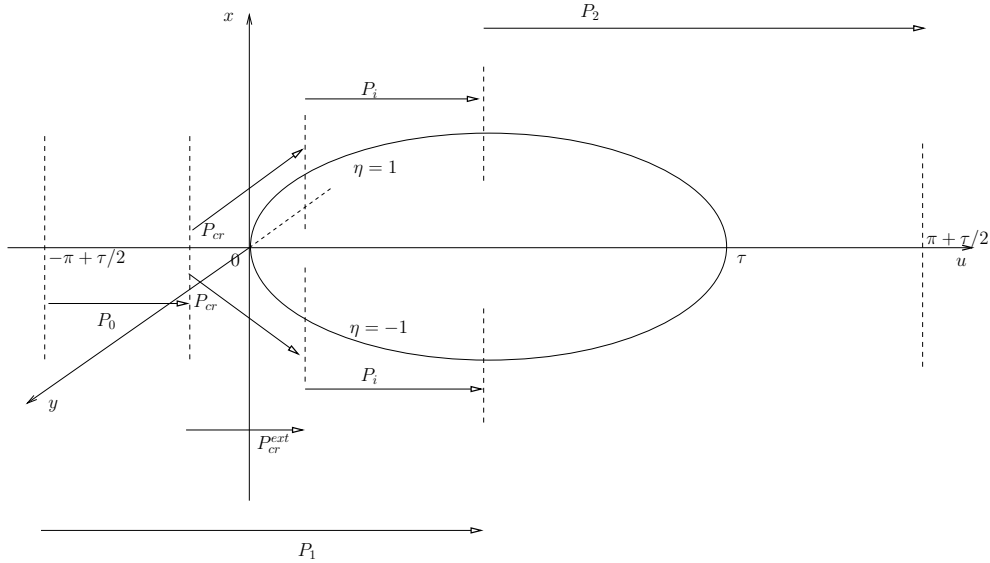


Figure 2: The return map P and its factors.

By assumption (A3) the equations of motion for $(x(u), y(u))$ are invariant with respect to the following transformation

$$\begin{aligned} x &\mapsto x, \\ y &\mapsto -y, \\ u &\mapsto \tau - u. \end{aligned}$$

This is just the time-reversible \mathcal{T}_τ -symmetry (3) viewed as an action on $(x, y) = (x, y)(u)$. From this follows:

Lemma 4. *The mapping P_2^{-1} is related to P_1 through the following expression:*

$$P_2^{-1} = E \circ P_1 \circ E,$$

where $E = \text{diag}(1, -1)$.

In the following I will obtain approximations to the inner and outer maps P_i and P_o , respectively. I start by considering the one that requires most effort, P_i .

5.1. The inner map P_i

To approximate the inner map we first apply averaging to (47). It will be important to keep track of how the frequency $\hat{\Omega} = \hat{\Omega}(\hat{u})$ enters the remainder, and to highlight the most important terms I will make use of the following two lemmata.

Lemma 5. *Let q and p be positive real numbers satisfying $0 < q < p$. Then there exists a constant $c = c(\hat{u}_*)$ so that*

$$\hat{\Omega}(\hat{u})^{-2p} \leq c^{p-q} \hat{\Omega}(\hat{u})^{-2q}, \quad (74)$$

for all $\hat{u} \in [\hat{u}_*, \mu^{-2}\tau/2]$.

PROOF. By choice of \hat{u}_* , the function $\hat{\Omega}(\hat{u})^{-2}$ is bounded from above by some constant $c = c(\hat{u}_*)$. Therefore

$$\frac{\hat{\Omega}(\hat{u})^{-2p}}{\hat{\Omega}(\hat{u})^{-2q}} = \hat{\Omega}(\hat{u})^{-2(p-q)} \leq c^{p-q},$$

given that $p > q$. □

Lemma 6. *Let $q \in \overline{\mathbb{R}}_+$. Given an integrable function $r = r(\hat{u})$, $\hat{u} \in [\hat{u}_*, \mu^{-2}\tau/2]$ satisfying the following estimate*

$$|r(\hat{u})| \leq \hat{\Omega}(\hat{u})^{-2q}, \quad \hat{u} \in [\hat{u}_*, \mu^{-2}\tau/2].$$

If $q < 1$ then there exists a $c_1 = c_1(q)$ so that

$$\left| \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} r(\hat{u}) d\hat{u} \right| \leq c_1 \mu^{-2(1-q)}.$$

If $q = 1$ then there exists a c_2 so that

$$\left| \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} r(\hat{u}) d\hat{u} \right| \leq c_2 \ln(\mu^{-2}\hat{u}_*^{-1}). \quad (75)$$

Finally if $q > 1$ then the corresponding integral is uniformly bounded with respect to ϵ : There exists a c_3 so that

$$\left| \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} r(\hat{u}) d\hat{u} \right| \leq c_3 (q-1)^{-1} \hat{u}_*^{1-q}.$$

PROOF. For $q \neq 1$ I have that

$$\begin{aligned}
\left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u}) d\hat{u} \right| &\leq \int_{\hat{u}_*}^{\mu^{-2}\tau/2} |r(\hat{u})| d\hat{u} \\
&\leq \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u})^{-2q} d\hat{u} \\
&= (\text{Use (45)}) \\
&\leq c_1 (\mu^2)^{q-1} \int_{u_*}^{\tau/2} \vartheta(u)^{-q} du,
\end{aligned}$$

setting $u_* = \mu^2 \hat{u}_*$. Here c_1 is some constant depending only q . I then use that $u\vartheta(u)^{-1} = 1 + \mathcal{O}(u)$ and therefore $\vartheta(u)^{-1} = u^{-1} + \mathcal{O}(1)$ cf. (46) for $u > 0$ to conclude

$$\left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u}) d\hat{u} \right| \leq c_2 c_1 \mu^{2(q-1)} (1-q)^{-1} ((\tau/2)^{1-q} - u_*^{1-q}), \quad (76)$$

for some constant c_2 again depending only on q . If $q < 1$ then $(\tau/2)^{1-q}$ dominates the last factor and I obtain the first result of the lemma:

$$\left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u}) d\hat{u} \right| \leq 2c_2 c_1 \mu^{-2(q-1)} (1-q)^{-1} (\tau/2)^{1-q},$$

for \hat{u}_* sufficiently small. For $q > 1$ the last term in (76) dominates for ϵ small and so upon inserting $u_* = \mu^2 \hat{u}_*$ I obtain

$$\left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u}) d\hat{u} \right| \leq 2c_2 c_1 (q-1)^{-1} \hat{u}_*^{1-q},$$

completing the third part for \hat{u}_* sufficiently small. The case $q = 1$ is also the consequence of a simple calculation. \square

Recall the form of $\hat{\Lambda}$ in (47):

$$\begin{aligned}
\hat{\Lambda} &= \zeta_0(\hat{u}, \hat{\nu}_0, \hat{\rho}_0) + \rho_0(\hat{u}, \hat{\rho}_0, \phi_0) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}), \\
\rho_0 &= \delta^{3/4} \hat{\Omega}(\hat{u})^{-1/2} \rho_{01} + \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \rho_{02}, \\
\hat{\omega} &= d\hat{\xi}_0 \wedge d\hat{\sigma}_0 + \delta^{-3/2} d\hat{u} \wedge d\hat{\nu}_0,
\end{aligned} \quad (77)$$

where I have used Lemma 5 to say that $\hat{\Omega}^{-5} \leq c^{3/4} \hat{\Omega}^{-7/2}$. In comparison with (47) I have also multiplied the symplectic form by μ (which corresponds to scaling time by μ). The averages of ρ_{01} and ρ_{02} in (49) and (50) are easily computed:

$$\begin{aligned}\bar{\rho}_{01}(u, \hat{\varrho}_0) &= \frac{1}{2\pi} \int_0^{2\pi} \rho_{01}(u, \hat{\varrho}_0, s) ds = 0, \\ \bar{\rho}_{02}(u, \hat{\varrho}_0) &= \frac{1}{2\pi} \int_0^{2\pi} \rho_{02}(u, \hat{\varrho}_0, s) ds = \frac{3}{4}(1 + \mathcal{O}(u))\hat{\varrho}_0^2.\end{aligned}$$

Therefore

$$\bar{\rho}_0(\hat{u}, \hat{\varrho}_0) = \frac{3}{4}\delta^{3/2}\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))\hat{\varrho}_0^2.$$

I also set $\tilde{\rho}_0 = \rho_0 - \bar{\rho}_0$ and $\tilde{\rho}_{02} = \rho_{02} - \bar{\rho}_{02}$ so that $\tilde{\rho}_0$ has zero average. I first realise the following:

Lemma 7. *The order of $\tilde{\rho}_0$ is $\hat{\Omega}^{-1/2}\delta^{3/4}$. Similarly the order of $\partial_{\hat{u}}\tilde{\rho}_0$ is $\hat{\Omega}^{-5/2}\delta^{3/4}$.*

PROOF. The term with ρ_{01} dominates the expression for ρ_0 cf. $\delta^{3/2} \ll \delta^{3/4}$ and using (74) to say that $\hat{\Omega}^{-2} \leq c^{3/4}\hat{\Omega}^{-1/2}$. This gives the first claim. The next claim follows from similar arguments upon differentiation with respect to \hat{u} :

$$\begin{aligned}\partial_{\hat{u}}\tilde{\rho}_0 &= -\frac{1}{2}\delta^{3/4}\hat{\Omega}(\hat{u})^{-5/2}(1 + \mathcal{O}(u))\rho_{01}(u, \hat{\varrho}_0, \phi_0) + \delta^{3/4}\mu^2\hat{\Omega}(\hat{u})^{-1/2}\partial_u\rho_{01}(u, \hat{\varrho}_0, \phi_0) \\ &\quad - 2\delta^{3/2}\hat{\Omega}(\hat{u})^{-4}(1 + \mathcal{O}(u))\tilde{\rho}_{02}(u, \hat{\varrho}_0, \phi_0) + \delta^{3/2}\mu^2\hat{\Omega}(\hat{u})^{-2}\partial_u\tilde{\rho}_{02}(u, \hat{\varrho}_0, \phi_0).\end{aligned}$$

Here I have used (51). Replacing μ^2 by (52) gives

$$\partial_{\hat{u}}\tilde{\rho} = \mathcal{O}(\hat{\Omega}^{-5/2}\delta^{3/4} + \hat{\Omega}^{-4}\delta^{3/2}) = \mathcal{O}(\hat{\Omega}^{-5/2}\delta^{3/4}),$$

which completes the result. \square

To push the phase-dependency to higher order, I then use the following generating function

$$G(\hat{u}, \hat{\nu}_1, \hat{\varrho}_0, \phi_1) = \delta^{-3/2}\hat{u}\hat{\nu}_1 + \hat{\varrho}_0\phi_1 + \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, s) ds,$$

to generate a transformation given as the solution to the equations:

$$\begin{aligned}\hat{u}_1 &= \delta^{3/2} \partial_{\hat{v}_1} G = \hat{u}, \\ \hat{v}_0 &= \delta^{3/2} \partial_{\hat{u}} G = \hat{v}_1 + \delta^{3/2} \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \partial_{\hat{u}} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, s) ds \\ &= \hat{v}_1 + \mathcal{O}(\hat{\Omega}(\hat{u})^{-7/2} \delta^{9/4}),\end{aligned}\tag{78}$$

using Lemma 7 in the last equality, and

$$\begin{aligned}\hat{\varrho}_1 &= \partial_{\phi_1} G = \hat{\varrho}_0 + \hat{\Omega}(\hat{u})^{-1} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1), \\ \phi_0 &= \partial_{\hat{\varrho}_0} G = \phi_1 + \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, s) ds.\end{aligned}$$

I then obtain the following system

$$\hat{\Lambda} = \zeta_1(\hat{u}, \hat{v}_1, \hat{\varrho}_1) + \rho_1(\hat{u}, \hat{v}_1, \hat{\varrho}_1) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}),$$

where

$$\begin{aligned}\zeta_1(\hat{u}, \hat{v}_1, \hat{\varrho}_1) &= \hat{v}_1 + \hat{\Omega}(\hat{u}) \hat{\varrho}_1 + \bar{\rho}_0(\hat{u}, \hat{\varrho}_1) = \hat{v}_1 + \hat{\Omega}(\hat{u}) \hat{\varrho}_1 + \frac{3}{4} \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{\varrho}_0^2, \\ \rho_1(\hat{u}, \hat{\varrho}_1, \phi_1) &= \bar{\rho}_0(\hat{u}, \hat{\varrho}_0) - \bar{\rho}_0(\hat{u}, \hat{\varrho}_1) + \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_0) - \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1) \\ &\quad + \delta^{3/2} \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \partial_{\hat{u}} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, s) ds.\end{aligned}$$

In the following lemma I estimate ρ_1 .

Lemma 8. *The remainder $\rho_1 = \rho_1(\hat{u}, \hat{\varrho}_1, \phi_1)$ takes the following form*

$$\rho_1(\hat{u}, \hat{\varrho}_1, \phi_1) = \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \partial_{\phi_0} \rho_{01}(u, \hat{\varrho}_1, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \rho_{01}(u, \hat{\varrho}_1, s) ds + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}).$$

PROOF. Firstly,

$$\begin{aligned}\bar{\rho}_0(\hat{u}, \hat{\varrho}_0) - \bar{\rho}_0(\hat{u}, \hat{\varrho}_1) &= \frac{3}{4} \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) (\hat{\varrho}_0 + \hat{\varrho}_1) (\hat{\varrho}_0 - \hat{\varrho}_1) \\ &= -\frac{3}{4} \delta^{3/2} \hat{\Omega}(\hat{u})^{-3} (1 + \mathcal{O}(u)) (\hat{\varrho}_0 + \hat{\varrho}_1) \tilde{\rho}(\hat{u}, \hat{\varrho}_0, \phi_1) \\ &= \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}),\end{aligned}$$

using Lemma 7. Next using Taylor's theorem:

$$\begin{aligned}
\tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_0) - \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1) &= \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1)(\phi_0 - \phi_1) + \int_0^1 (1-s) \\
&\quad \times \partial_{\phi_0}^2 \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1 + s(\phi_0 - \phi_1))(\phi_0 - \phi_1)^2 ds \\
&= \hat{\Omega}(\hat{u})^{-1} \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, s) ds + \mathcal{O}(\hat{\Omega}(\hat{u})^{-7/2} \delta^{9/4}) \\
&= \hat{\Omega}(\hat{u})^{-1} \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, s) ds + \mathcal{O}(\hat{\Omega}(\hat{u})^{-7/2} \delta^{9/4}),
\end{aligned}$$

having here also used that $\hat{\varrho}_0 - \hat{\varrho}_1, \phi_0 - \phi_1 = \mathcal{O}(\hat{\Omega}(\hat{u})^{-3/2} \delta^{3/4})$ cf. Lemma 7. I complete the result by using (78) and the following:

$$\begin{aligned}
\hat{\Omega}(\hat{u})^{-1} \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, s) ds &= \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \partial_{\phi_0} \rho_{01}(u, \hat{\varrho}_1, \tau) \\
&\quad \times \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \rho_{01}(u, \hat{\varrho}_1, s) ds + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}),
\end{aligned}$$

which follows from (77). \square

One more averaging step is needed to push the order of the error below $\hat{\Omega}^{-2} \delta^{3/2}$: Note that its contribution matters cf. (75) on the time scale $u \in [\mu^2 \hat{u}, \tau/2]$ relevant for P_i , see (73). I therefore define

$$\rho_{12}(u, \hat{\varrho}_1, \phi_1) = \partial_{\phi_0} \rho_{01}(u, \hat{\varrho}_1, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \rho_{01}(u, \hat{\varrho}_1, s) ds,$$

so that

$$\rho_1(\hat{u}, \hat{\varrho}_1, \phi_1) = \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \rho_{12}(u, \hat{\varrho}_1, \phi_1) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}).$$

The average of ρ_{12} is easily computed given (49):

$$\bar{\rho}_{12}(u, \hat{\varrho}_1) = -\frac{15}{4}(1 + \mathcal{O}(u)) \hat{\varrho}_1^2 - \frac{1}{8}(1 + \mathcal{O}(u)).$$

I set $\tilde{\rho}_1 = \rho_1 - \bar{\rho}_1$ and use the following generating function

$$G(\hat{u}_1, \hat{\nu}, \hat{\varrho}_1, \phi) = \delta^{-3/2} \hat{u}_1 \hat{\nu} + \hat{\varrho}_1 \phi + \hat{\Omega}(\hat{u}_1)^{-1} \int_0^\phi \tilde{\rho}_1(\hat{u}_1, \hat{\varrho}_1, s) ds,$$

to generate a final transformation

$$(\hat{u}_1, \hat{\nu}_1, \hat{\varrho}_1, \phi_1) \mapsto (\hat{u}, \hat{\nu}, \hat{\varrho}, \phi).$$

This gives the following form of the Hamiltonian in the new variables:

$$\hat{\Lambda} = \zeta(\hat{u}, \hat{\nu}, \hat{\varrho}) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}),$$

where

$$\zeta(\hat{u}, \hat{\nu}, \hat{\varrho}) = h_1(\hat{u}, \hat{\nu}, \hat{\varrho}) + \bar{\rho}_1(\hat{u}, \hat{\varrho}) = \hat{\nu} + \hat{\Omega}(\hat{u})\hat{\varrho} - 3\delta^{3/2}\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))\hat{\varrho}^2.$$

I have here ignored the term $-\frac{1}{8}\delta^{3/2}\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))$ that only depends on u ; it can be removed by a further translation of $\hat{\nu}$. The equations of motion are

$$\frac{d\hat{\varrho}}{d\hat{u}} = \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{3/4}), \quad (79)$$

$$\frac{d\phi}{d\hat{u}} = -\delta^{-3/2}\hat{\Omega}(\hat{u}) + 6\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))\hat{\varrho} + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{3/4}). \quad (80)$$

The mapping P_i describes the assignment $(\hat{\varrho}_0, \phi_0)(\hat{u} = \hat{u}_*) \mapsto (\hat{\varrho}_0, \phi_0)(\hat{u} = \mu^{-2}\tau/2)$. To approximate this I will solve the truncation of the transformed differential equations (79) and (80):

$$\begin{aligned} \frac{d\hat{\varrho}}{d\hat{u}} &= 0, \\ \frac{d\phi}{d\hat{u}} &= -\delta^{-3/2}\hat{\Omega}(\hat{u}) + 6\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))\hat{\varrho}, \end{aligned}$$

from $\hat{u} = \hat{u}_*$ to $\hat{u} = \mu^{-2}\tau/2$. This is adequate because of the following:

Lemma 9. *The action $\hat{\varrho}_0$ is conserved on the interval from $\hat{u} = \hat{u}_*$ to $\hat{u} = \mu^{-2}\tau/2$ with an accuracy of $\delta^{3/4}$.*

PROOF. I use Lemma 6 with $|r(\hat{u})| \leq \hat{\Omega}^{-7/2}$ ($q = 7/4 > 1$) together with (79) to conclude that the variation $\Delta\hat{\varrho}$ of $\hat{\varrho}$ on the given interval can be estimated from above by

$$|\Delta\hat{\varrho}| \leq c\delta^{3/4}.$$

Since $\hat{\varrho} - \hat{\varrho}_0 = \mathcal{O}(\delta^{3/4})$ this completes the result. \square

By a similar argument, I estimate the effect of the remainder in (80) by $\mathcal{O}(\delta^{3/4})$ and I compute the variation in ϕ by

$$\begin{aligned} \phi(\epsilon^{-2/3}\delta\tau/2) &= \phi(\hat{u}_*) - \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \left(\delta^{-3/2}\hat{\Omega}(\hat{u}) - 6\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u)) \right) \hat{\varrho}_0 \\ &\quad + \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \mathcal{O}(\hat{\Omega}(\hat{u})^{-2}\delta^{3/4}) d\hat{u} + \mathcal{O}(\delta^{3/4}). \end{aligned} \quad (81)$$

The remainder $\mathcal{O}(\hat{\Omega}(\hat{u})^{-2}\delta^{3/4})$ in the integral in (81) comes from $\hat{\varrho}(\hat{u}) = \hat{\varrho}_0 + \mathcal{O}(\delta^{3/4})$ with $\hat{\varrho}_0 = \text{const.}$ on this interval. This term can be estimated from above by a term of order $\delta^{3/4} \ln \epsilon^{-1}$ cf. Lemma 6 with $q = 1$. The following lemma gives asymptotics of the two other integrals appearing in (81).

Lemma 10.

$$\delta^{-3/2} \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u}) d\hat{u} = \sqrt{2}\epsilon^{-1}e_1 - \frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}_*^{3/2} + \mathcal{O}(\epsilon^{2/3}\delta^{-5/2}).$$

with

$$e_1 = \int_0^{\tau/2} \vartheta(u)^{1/2} du,$$

with $\vartheta = \vartheta(u) = u + \mathcal{O}(u^2)$ (46). Moreover, there exists a positive constant e_2 independent of ϵ such that

$$\int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u)) d\hat{u} = \frac{1}{3} \ln(e_2\epsilon^{-1}) - \frac{1}{2} \ln(\delta^{-1}\hat{u}_*) + \mathcal{O}(\mu^2).$$

PROOF. I use (46) to write $\hat{\Omega}(\hat{u})$ as $\sqrt{2}\mu^{-1}\vartheta(u)^{1/2}$ with $\vartheta(0) = 0$, $\vartheta'(0) = 1$. Therefore

$$\begin{aligned} \delta^{-3/2} \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u}) d\hat{u} &= \sqrt{2}\epsilon^{-1} \int_{u_*}^{\tau/2} \vartheta(u)^{1/2} du \\ &= \sqrt{2}\epsilon^{-1}e_1 - \sqrt{2}\epsilon^{-1} \int_0^1 \vartheta(u_*s)^{1/2} ds u_*, \end{aligned} \quad (82)$$

here $u_* = \mu^2 \hat{u}_*$. For the last integral I use the following

$$\begin{aligned} \int_0^1 u_*^{-1/2} \vartheta(u_* s)^{1/2} ds u_*^{3/2} &= \int_0^1 (1 + \mathcal{O}(u_*^{3/2} s^{3/2})) ds u_*^{3/2} \\ &= \frac{2}{3} u_*^{3/2} + \mathcal{O}(u_*^{5/2}). \end{aligned} \quad (83)$$

Inserting this back into (82) completes the first part of the proof.

For the second part, the integral is written as

$$\int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) d\hat{u} = \frac{1}{2} \int_{u_*}^{\tau/2} \vartheta(u)^{-1} (1 + \mathcal{O}(u)) du.$$

I write $\vartheta(u)^{-1} = u^{-1} + \mathcal{O}(1)$ for u small so that

$$\begin{aligned} \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) d\hat{u} &= \frac{1}{2} \int_{u_*}^{\tau/2} u^{-1} du + \mathcal{O}(1) \\ &= \frac{1}{2} \ln(\mu^{-2} \hat{u}_*^{-1}) + \mathcal{O}(1) \\ &= \frac{1}{3} \ln \epsilon^{-1} - \frac{1}{2} \ln(\delta^{-1} \hat{u}_*) + \mathcal{O}(1), \end{aligned}$$

using that $\mu^2 = \epsilon^{2/3} \delta^{-1}$. The $\mathcal{O}(1)$ -term is smooth as a function of u_* and can therefore be written as $\text{const.} + \mathcal{O}(u_*)$ for u_* small by Taylor's theorem. Writing the constant as $\frac{1}{3} \ln e_2$ completes the proof. \square

Following this lemma I can write (81) as

$$\begin{aligned} \phi_0(\mu^{-2}\tau/2) &= \phi_0(\hat{u}_*) - \sqrt{2}\epsilon^{-1}e_1 + 2\ln(e_2\epsilon^{-1})\hat{\varrho}_0 + \frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}_*^{3/2} - 3\ln(\delta^{-1}\hat{u}_*)\hat{\varrho}_0 \\ &\quad + \mathcal{O}(\delta^{3/4} \ln \epsilon^{-1}), \end{aligned}$$

using $\phi_0 = \phi + \mathcal{O}(\delta^{3/4})$. I collect the result about P_i in the following proposition:

Proposition 2. *Suppose that*

$$\delta^{3/4} \ln \epsilon^{-1} \ll 1, \quad (84)$$

then the mapping P_i , see (73), satisfies

$$\begin{aligned} P_i(\hat{\varrho}_0, \phi_0) &= \left(\hat{\varrho}_0 \right. \\ &\quad \left. \phi_0 - \sqrt{2}\epsilon^{-1}e_1 + (2\ln(e_2\epsilon^{-1}) - 3\ln(\delta^{-1}\hat{u}_*))\hat{\varrho}_0 + \frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}_*^{3/2} \right) \\ &\quad + \mathcal{O}(\delta^{3/4} \ln \epsilon^{-1}). \end{aligned} \quad (85)$$

5.2. The outer map P_o

The derivation of an approximation of the outer map P_o in (72) is similar to the derivation presented above in Section 5.1 for the inner map P_i . The details are therefore delayed to Appendix B and I simply state the result:

Proposition 3. *Suppose that*

$$\delta^{3/2} \ln \epsilon^{-1} \ll 1,$$

then

$$\begin{aligned} P_o(\hat{z}_0, w_0) = & \left(w_0 - \epsilon^{-1} e_3 - (\ln(e_4 \epsilon^{-1}) - \frac{3}{2} \ln(\delta^{-1} \hat{u}_*)) \hat{z}_0 + \frac{2}{3} \delta^{-3/2} \hat{u}_*^{3/2} \right) \\ & + \mathcal{O}(\delta^{3/2} \ln \epsilon^{-1}). \end{aligned} \quad (86)$$

Here

$$e_3 = \int_0^{\pi-\tau/2} (-f(-u))^{1/2} du,$$

while e_4 is another positive constant defined in Appendix B.

Remark 9. The condition $\delta^{3/2} \ln \epsilon^{-1} \ll 1$ in Proposition 3 is weaker than the condition $\delta^{3/4} \ln \epsilon^{-1} \ll 1$ in Proposition 2. Note that it is possible to realize $\delta^{3/4} \ln \epsilon^{-1} \ll 1$ without violating condition (69). \diamond

5.3. Fix point Eq.

To obtain periodic orbits I solve the following fix point equation

$$\chi(\hat{z}_0, w_0) = P(\hat{z}_0, w_0), \quad P = P_2 \circ P_1, \quad (87)$$

up to symmetry $\chi \in \{Id, \mathcal{R}\}$. Here $\{Id, \mathcal{R}\}$ is the group generated by the translation \mathcal{R} from (64). A simple calculation gives the following result:

Lemma 11. *If $\chi = \mathcal{R}$ in (87) then (\hat{z}_0, w_0) is a periodic-2 point.*

PROOF. It follows from the fact that $\mathcal{R}^2 = Id$ and that P is equivariant with respect to the action of \mathcal{R} . \square

To avoid having to invert the crossing map $P_{\text{cr}}^{\text{ext}}$ I re-write (87) as

$$P_1 \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} = P_2^{-1} \circ \chi \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} = E \circ P_1 \circ E \circ \chi \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix}, \quad E = \text{diag}(1, -1), \quad (88)$$

by inverting P_2 using Lemma 4. The mapping P_1 is then replaced by

$$P_i \circ P_{\text{cr}}^{\text{ext}} \circ P_o,$$

and then by using the equivariance of P_o and E with respect to the \mathcal{R} -action, one easily verifies, that (88) implies that

$$P_i \circ P_{\text{cr}} \circ P_o \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} = E \circ P_i \circ P_{\text{cr}} \circ P_o \circ E \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix}, \quad (89)$$

taking $\chi = Id$ if

$$\eta \left(P_o \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} \right) = \eta \left(P_o \circ E \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} \right),$$

and taking, cf. (66), $\chi = \mathcal{R}$, if this last equality, on the other hand, does not hold.

Remark 10. A solution (\hat{z}_0, w_0) to (89) always defines a periodic orbit. Solutions with $\chi = Id$ correspond to fix points of P , that is periodic orbits of (12) with periods $T = 2\pi\epsilon^{-1}$, where $(|x(u)|, y(u))$ remains close to the singular solution $(|x_s(u)|, y_s)$ (9). The periodic-2 points, that appear when one has to take χ to be \mathcal{R} , are fix points of P^2 and correspond to periodic orbits of twice the period: $T = 4\pi\epsilon^{-1}$. Here $(|x(u)|, y(u))$ is still close to the singular solution $(|x_s(u)|, y_s)$ (9), but in this case the motion alternates between being close to $\kappa(u)$ to being close to $-\kappa(u)$: It is \mathcal{R} -symmetric. The latter property is a consequence of the fact that if w_0 is shifted by π then λ is also shifted by π , cf. (55) and (61). By definition this changes the sign of η and what route is followed on S . The symmetry properties of the periodic orbits are also the subject of Proposition 5 below. \diamond

6. Solving the fix point Eq. (89) - Proof of the main results

In the following two sections, the two sides of Eq. (89) are computed using the approximations of P_o , P_{cr} and P_i established above.

The left hand side of Eq. (89)

Setting $w_0(-\hat{u}_*)$ from (86) equal to the $w_0(-\hat{u}_*)$ in (55) it is realized that the phase $l = l_l$ in (55), using here the subscript l to indicate “left”, is given as

$$l_l = w_0 - \epsilon^{-1}e_3 - \ln(e_4\epsilon^{-1})\hat{z}_0 + \frac{\pi}{2}, \quad (90)$$

ignoring for simplicity the $\mathcal{O}(\delta^{3/2} \ln \epsilon^{-1})$ -remainder. The image $P_{\text{cr}} \circ P_o(\hat{z}_0, w_0)$ therefore becomes

$$\begin{aligned} \hat{\varrho}_0 &= \frac{1}{2\pi} \ln \frac{1 + |p_l|^2}{2|\text{Im } p_l|}, \\ \phi_0 &= -\frac{2\sqrt{2}}{3} \delta^{-3/2} \hat{u}_*^{3/2} + 3 \ln(\delta^{-1} \hat{u}_*) \hat{\varrho}_0 - \theta_l, \end{aligned} \quad (91)$$

where

$$\theta_l = \theta(\hat{\varrho}_0, p_l) \text{ with } \theta \text{ as in (58),} \quad (92)$$

$p_l = p(\hat{z}_0, \lambda_l)$ from (60) and

$$\lambda_l = 3\hat{z}_0 \ln 2 - \frac{\pi}{4} - \arg \Gamma(i\hat{z}_0) - l_l.$$

Given (90) it follows that $\lambda_l(\hat{z}_0, w_0)$ can be written as

$$\lambda_l = -w_0 + \ln(e_4\epsilon^{-1})\hat{z}_0 + G(\hat{z}_0), \quad (93)$$

$$G(\hat{z}_0) = 3\hat{z}_0 \ln 2 - \frac{3\pi}{4} - \arg \Gamma(i\hat{z}_0) + \epsilon^{-1}e_3. \quad (94)$$

Finally applying P_i in (85) to the Eqs. (91) gives the left hand side of (89) which is denoted by $(\hat{\varrho}_l, \phi_l)$:

$$\hat{\varrho}_l = \frac{1}{2\pi} \ln \frac{1 + |p_l|^2}{2|\text{Im } p_l|}, \quad (95)$$

$$\phi_l = -\sqrt{2}\epsilon^{-1}e_1 + 2 \ln(e_2\epsilon^{-1})\hat{\varrho}_l - \theta_l. \quad (96)$$

ignoring the $\mathcal{O}(\delta^{3/2} \ln \epsilon^{-1})$ -remainder.

The right hand side of Eq. (89)

For the right hand side, the image of $P_o \circ E$, $(\hat{z}_0, w_0)(\mu^{-2}\tau + \hat{u}_*) = P_o \circ E(\hat{z}_0, w_0)$, is initially computed using (86):

$$\begin{aligned}\hat{z}_0(\mu^{-2}\tau + \hat{u}_*) &= \hat{z}_0, \\ w_0(\mu^{-2}\tau + \hat{u}_*) &= -w_0 - \epsilon^{-1}e_3 - \ln(e_4\epsilon^{-1})\hat{z}_0 + \frac{2}{3}\delta^{-3/2}\hat{u}_*^{3/2} + \frac{3}{2}\ln(\delta^{-1}\hat{u}_*).\end{aligned}\tag{97}$$

Here the $\mathcal{O}(\delta^{3/2}\ln\epsilon^{-1})$ -remainder has again been left out for simplicity. The phase $w_0(\mu^{-2}\tau + \hat{u}_*)$ is then, much as above, set equal to the phase w_0 in (55). This gives the following expression for the phase $l = l_r$, now using the subscript r to indicate “right”:

$$l_r = -w_0 - \epsilon^{-1}e_3 - \ln(e_4\epsilon^{-1})\hat{z}_0 + \frac{\pi}{2}.\tag{98}$$

Then upon applying P_{cr} to (97) I obtain

$$\begin{aligned}\hat{\varrho}_0 &= \frac{1}{2\pi} \ln \frac{1 + |p_r|^2}{2|\text{Im } p_r|}, \\ \phi_0 &= -\frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}_*^{3/2} + 3\ln(\delta^{-1}\hat{u}_*)\hat{\varrho}_0 - \theta_r,\end{aligned}$$

with $\theta_r = \theta(\hat{\varrho}_0, p_r)$ from (58), $p_r = p(\hat{z}_0, \lambda_r)$ as in (60) and

$$\lambda_r = 3\hat{z}_0 \ln 2 - \frac{\pi}{4} - \arg \Gamma(i\hat{z}_0) - l_r.$$

Given (98) it follows that $\lambda_r = \lambda_r(\hat{z}_0, w_0)$ takes the following form:

$$\lambda_r = w_0 + \ln(e_4\epsilon^{-1})\hat{z}_0 + G(\hat{z}_0),\tag{99}$$

with G as in (94). Finally, $E \circ P_i$ is applied to this, using the approximation (85) for P_i . This finally gives the right hand side of (89) which is denoted by $(\hat{\varrho}_r, \phi_r)$

$$\hat{\varrho}_r = \frac{1}{2\pi} \ln \frac{1 + |p_r|^2}{2|\text{Im } p_r|},\tag{100}$$

$$\phi_r = \sqrt{2}\epsilon^{-1}e_1 - 2\ln(e_2\epsilon^{-1})\hat{\varrho}_r + \theta_r.\tag{101}$$

ignoring the $\mathcal{O}(\delta^{3/2}\ln\epsilon^{-1})$ -remainder. The equation (89) therefore becomes $\hat{\varrho}_l(\hat{z}_0, w_0) = \hat{\varrho}_r(\hat{z}_0, w_0)$, $\phi_l(\hat{z}_0, w_0) = \phi_r(\hat{z}_0, w_0)$, see (95), (96), (100), and (101).

Solving $\hat{\varrho}_l = \hat{\varrho}_r$ using (95) and (100)

The absolute value of p depends, cf. (60), only on \hat{z}_0 . Hence $|p_l| = |p_r|$. Setting

$$\hat{\varrho}_0 \equiv \hat{\varrho}_l(\hat{z}_0, w_0) = \hat{\varrho}_r(\hat{z}_0, w_0),$$

I therefore conclude that

$$\sin \lambda_l = \pm \sin \lambda_r, \quad (102)$$

and hence

$$\lambda_r = \begin{cases} \pm \lambda_l \\ \pi \pm \lambda_l \end{cases}, \quad (103)$$

with $\lambda_l(\hat{z}_0, w_0)$ and $\lambda_r(\hat{z}_0, w_0)$ given by Eqs. (93) and (99), respectively. Before continuing with solving the equation:

$$\phi_l = \phi_r \mod 2\pi,$$

I first compute the trace of the Jacobian matrix $\partial P = \partial_{(\hat{z}_0, w_0)} P$ of the truncation of the mapping. For this Eq. (103) will be used. I compute this trace for two reasons: (i) To decide when the contraction mapping theorem can be applied to conclude that the solutions can be continued into true solutions of the non-truncated equations. (ii): To investigate the stability of the periodic orbits.

The Jacobian of P

The function η is locally constant so it can be ignored completely in the calculations.

Lemma 12. *Suppose $1 + p^2 \neq 0$. If (i) $\lambda_r = \lambda_l$ or $\pi + \lambda_l$ then*

$$\begin{aligned} \text{tr}(\partial P) = 2 + 4B & \left((2 \ln(e_2 \epsilon^{-1}) - q)(\ln(e_4 \epsilon^{-1}) + g)B \right. \\ & \left. + A(2 \ln(e_2 \epsilon^{-1}) - q) + D(\ln(e_4 \epsilon^{-1}) + g) + C \right). \end{aligned} \quad (104)$$

Here

$$g = g(\hat{z}_0) = \partial_{\hat{z}_0} G = 3 \ln 2 - \partial_{\hat{z}_0} \arg \Gamma(i \hat{z}_0), \quad (105)$$

and

$$q = q(\hat{\varrho}_0) = \partial_{\hat{\varrho}_0} Q = 7 \ln 2 - \partial_{\hat{\varrho}_0} \arg \Gamma(2i\hat{\varrho}_0), \quad (106)$$

are the derivatives of $G = G(\hat{z}_0)$ (94) and $Q = Q(\hat{\varrho}_0)$ (59) with respect to \hat{z}_0 and $\hat{\varrho}_0$, respectively. Moreover

$$A = \partial_{\hat{z}_0} \varrho_0(\hat{z}_0, \lambda) = \frac{|p|^2 + 1}{2|p|^2}, \quad (107)$$

$$B = \partial_{\lambda} \varrho_0(\hat{z}_0, \lambda) = -\frac{1}{2\pi} \cot \lambda, \quad (108)$$

with ϱ_0 from (56), and

$$C = \partial_{\hat{z}_0} (\arg(1 + p^2)) = \frac{2\pi(1 + |p|^2) \sin(2\lambda)}{|1 + p^2|^2}, \quad (109)$$

$$D = \partial_{\lambda} (\arg(1 + p^2)) = \frac{2|p|^2 (\cos(2\lambda) + |p|^2)}{|1 + p^2|^2}. \quad (110)$$

These functions are all evaluated for $p = p_l(\hat{z}_0, \lambda)$ and $\lambda = \lambda_l(\hat{z}_0, w_0)$ in (104). On the other hand, if (ii) $\lambda_r = -\lambda_l$ or $\pi - \lambda_l$ then

$$\text{tr}(\partial P) = 2 - 4(2 \ln(e_2 \epsilon^{-1}) - g)(\ln(e_4 \epsilon^{-1}) + q) B^2, \quad (111)$$

with B (108) evaluated at $\lambda = \lambda_l(\hat{z}_0, w_0)$.

Remark 11. Note that the lemma, cf. (103), considers all of the four different scenarios. Note also that $A = A(\hat{z}_0)$ in (107) only depends upon \hat{z}_0 since $|p|$, cf. (60), is independent of λ \diamond

PROOF. I use the expressions for $(\hat{\varrho}_l, \phi_l)$ in (95) and (96) and $(\hat{\varrho}_r, \phi_r)$ in (100) and (101). For case (ii) I also use that the functions A and D are even with respect to λ . The functions B and C are, on the other hand, odd. \square

Eqs. (104) and (111) are asymptotically of the form:

$$\text{tr}(\partial P) - 2 = \mp 8 \ln^2(\epsilon^{-1}) B^2 + \mathcal{O}(\ln \epsilon^{-1}), \quad (112)$$

respectively. Therefore

Lemma 13. *Consider all solutions of (87) with λ_l belonging to $[c^{-1}, \pi/2 - c^{-1}] \cup [\pi/2 + c^{-1}, \pi - c^{-1}]$ or a π -translation of this set. These periodic orbits are all unstable for ϵ sufficiently small.*

PROOF. A sufficient condition for instability is that

$$|\text{tr } \partial P| > 2, \quad (113)$$

using also here that $\text{tr } (\partial P)^2 = \text{tr}^2 \partial P - 2$. For the considered λ_l -values $|B| \geq c_1^{-1} > 0$, c_1 independent of ϵ , and so (113) can always be archived by taking ϵ sufficiently small cf. (112). \square

Stable orbits can therefore only occur if λ_l is near $\pi/2$ or $3\pi/2$ where B (108) is small. I consider this in Section 6.3.

6.1. Unstable solutions - Part 1° of the main result

In this section I will find unstable periodic orbits. I continue from (103) and start by dividing the presentation into two separate cases:

- case (i) where $\lambda_r = \lambda_l$ or $\lambda_r = \pi + \lambda_l$;
- case (ii) where $\lambda_r = -\lambda_l$ or $\lambda_r = \pi - \lambda_l$.

These two cases cover all the possible scenarios. They also correspond to solutions with different symmetry properties (see Section 6.2 below).

Consider first case (i) and $\lambda_r = \lambda_l$. Then by using (93) and (99)

$$w_0 = 0 \quad \text{or} \quad \pi.$$

If $\lambda_r = \pi + \lambda_l$ then

$$w_0 = \pi/2 \quad \text{or} \quad 3\pi/2.$$

It follows that if

$$w_0 = 0 \quad \text{or} \quad \pi/2 \pmod{\pi},$$

then $\lambda_r = \lambda_l \pmod{\pi}$. Also by (60)

$$p_r = \pm p_l. \quad (114)$$

On the other hand, for case (ii) with $\lambda_r = -\lambda_l$ then

$$2 \ln(e_4 \epsilon^{-1}) \hat{z}_0 + 2G(\hat{z}_0) = 0 \pmod{2\pi},$$

using (93) and (99). For $\lambda_r = \pi - \lambda_l$ then I similarly get

$$2 \ln(e_4 \epsilon^{-1}) \hat{z}_0 + 2G(\hat{z}_0) = \pi \pmod{2\pi}.$$

It therefore follows that solutions to the equation

$$2 \ln(e_4 \epsilon^{-1}) \hat{z}_0 + 2G(\hat{z}_0) = 0 \pmod{\pi},$$

solve $\lambda_r = -\lambda_l \pmod{\pi}$. Also

$$p_r = \pm \bar{p}_l, \tag{115}$$

cf. (60).

Next, I consider the equation $\phi_l = \phi_r$ and use (96) and (101). In case (i) I use (114) so that $\theta_r = \theta_l$, $\theta = \theta(\hat{\varrho}_0, p)$ (58) depending only on p^2 , and conclude that

$$-\sqrt{2}\epsilon^{-1}e_1 + 2 \ln(e_2 \epsilon^{-1}) \hat{\varrho}_0 - \theta_l = 0 \pmod{\pi},$$

with $\theta_l = \theta(\hat{\varrho}_0, p_l)$, $p_l = (e^{2\pi\hat{z}_0} - 1)^{1/2} e^{i\lambda_l}$.

For (ii) I similarly use (115) so that $\theta_r = Q - \arg(1 + p_r^2) = Q + \arg(1 + p_l^2)$ and therefore

$$-\sqrt{2}\epsilon^{-1}e_1 + 2 \ln(e_2 \epsilon^{-1}) \hat{\varrho}_0 - Q = 0 \pmod{\pi},$$

with $Q = Q(\hat{\varrho}_0)$ (59). I collect the results in the following proposition:

Proposition 4. *Periodic orbits can be obtained as*

(i)

$$w_0 = 0, \pi/2 \pmod{\pi}, \tag{116}$$

$$F_i^{(2)}(\hat{z}_0) = 0 \pmod{\pi}, \tag{117}$$

where

$$F_i^{(2)}(\hat{z}_0) \equiv -\sqrt{2}\epsilon^{-1}e_1 + 2 \ln(e_2 \epsilon^{-1}) \hat{\varrho}_0(\hat{z}_0, \lambda_l(\hat{z}_0)) - \theta_l(\hat{z}_0), \tag{118}$$

with θ_l from (92), λ_l as in (93) here both viewed as a function of \hat{z}_0 only, given here that the value of $w_0 \pmod{\pi/2}$ is fixed by (116).

(ii)

$$F_{ii}^{(1)}(\hat{z}_0) = 0 \mod \pi, \quad (119)$$

$$F_{ii}^{(2)}(\hat{z}_0, \lambda_l) = 0 \mod \pi, \quad (120)$$

where

$$\begin{aligned} F_{ii}^{(1)}(\hat{z}_0) &\equiv 2 \ln(e_4 \epsilon^{-1}) \hat{z}_0 + 2G(\hat{z}_0), \\ F_{ii}^{(2)}(\hat{z}_0, \lambda) &\equiv -\sqrt{2} \epsilon^{-1} e_1 + 2 \ln(e_2 \epsilon^{-1}) \hat{\rho}_0 - Q(\hat{z}_0). \end{aligned}$$

The action $\hat{\rho}_0$ is given by (95):

$$\hat{\rho}_0 = \frac{1}{2\pi} \ln \frac{1 + |p_l|^2}{2|\operatorname{Im} p_l|} = \frac{\hat{z}_0}{2} + \frac{1}{4\pi} \ln (1 - e^{-2\pi \hat{z}_0})^{-1} - \frac{1}{2\pi} \ln(2|\sin \lambda_l|). \quad (121)$$

Here $\lambda_l = \lambda_l(\hat{z}_0, w_0)$ from (93).

I will address symmetry properties in the following section.

6.2. Symmetry properties

To any solution $(x, y)(u)$ there exists two solutions obtained by applying the symmetry \mathcal{R} and the time-reversible symmetry (A3) \mathcal{T}_τ :

$$\mathcal{R}(x, y, u)(t) = (-x, -y, u)(t), \quad \mathcal{T}_\tau(x, y, u)(t) = (x, -y, \tau - u)(-t).$$

Proposition 5. *Generic periodic orbits obtained from case (i) corresponds orbits that are symmetric with respect to \mathcal{R} and \mathcal{T}_τ ($4\pi\epsilon^{-1}$ -periodic) or just \mathcal{T}_τ ($2\pi\epsilon^{-1}$ -periodic). Case (ii) consists of non-symmetric $2\pi\epsilon^{-1}$ -periodic orbits or \mathcal{R} -symmetric $4\pi\epsilon^{-1}$ -periodic orbits.*

PROOF. Take $t = 0$ to be the time when $u = -\pi + \tau/2$. In case (i): $w_0 = 0, \pi/2 \mod \pi$ up to an error $\mathcal{O}(\delta^{3/4} \ln \epsilon^{-1})$. This means that either $x(0) = 0$ or $y(0) = 0$ at $t = 0$. Consider a periodic orbit with $y(0) = 0$ first. This is the case when $\lambda_r = \lambda_l$ so the periodic orbit is a fix point of P . The initial condition is near $(x(0), 0, -\pi + \tau/2)$. The initial condition for the solution

$$\mathcal{T}_\tau(x, y, u) = (x, -y, \tau - u)(-t),$$

is near

$$(x(0), 0, \tau - (-\pi + \tau/2)) = (x(0), 0, \pi + \tau/2) = (x(0), 0, -\pi + \tau/2).$$

By local uniqueness of generic periodic orbits it follows that the periodic orbit coincides with its symmetry-related \mathcal{T}_τ -periodic orbit.

Consider next $x(0) = 0$. This is the case when $\lambda_r = \pi + \lambda_l$ so the periodic orbit is a fix point of P^2 . The initial condition is near $(0, y(0), -\pi + \tau/2)$. The image of P is near $(0, -y(0), -\pi + \tau/2)$. The initial condition for the solution $\mathcal{R}(x(t), y(t), u(t)) = (-x(t), -y(t), u(t))$ is near

$$(0, -y(0), -\pi + \tau/2).$$

Similarly the initial condition for the solution

$$\mathcal{T}_\tau(x(t), y(t), u(t)) = (x(-t), -y(-t), \tau - u(-t)),$$

is near

$$(0, -y(0), \tau - (-\pi + \tau/2)) = (0, -y(0), \pi + \tau/2) = (0, -y(0), -\pi + \tau/2).$$

By local uniqueness of generic periodic orbits it follows that the periodic orbit is \mathcal{R} and \mathcal{T}_τ -symmetric. The argument for the \mathcal{R} -symmetry can be modified so that it also applies to the $4\pi\epsilon^{-1}$ -periodic orbits in case (ii). \square

Corollary 1. *Generic symmetric periodic orbits from case (i) satisfy the \mathcal{T}_τ -symmetry condition*

$$(x, y, u)(t) = (x, -y, \tau - u)(-t),$$

when they are $2\pi\epsilon^{-1}$ -periodic, or a translated version

$$(x, y, u)(t + 2\pi\epsilon^{-1}) = (x, -y, \tau - u)(-t),$$

when they are $4\pi\epsilon^{-1}$ -periodic. In the latter case, the following \mathcal{R} -symmetry condition also applies

$$(x, y, u)(t + 2\pi\epsilon^{-1}) = (-x, -y, u)(t),$$

Note that these conditions imply that $y = 0$ at $u = \tau/2$ in agreement with the fact that (117) is just the condition that $\phi_l = 0 \bmod \pi$ cf. (96).

I then show the following:

Theorem 14. Consider $\hat{z}_0 \in [c_1^{-1}, c_2]$ and $\lambda_l(\hat{z}_0, w_0) \in [c^{-1}, \pi - c^{-1}] \cup [\pi + c^{-1}, 2\pi - c^{-1}]$. Then for both cases (i) (\mathcal{R} and \mathcal{T}_τ -symmetric if they have period $4\pi\epsilon^{-1}$ or just \mathcal{T}_τ if they have period $2\pi\epsilon^{-1}$) and (ii) (\mathcal{R} -symmetric or non-symmetric) the following statement holds true: For ϵ sufficiently small there exist $\mathcal{O}(\ln^2 \epsilon^{-1})$ -many unstable periodic orbits. The characteristic multipliers are $\mathcal{O}(\ln^{\pm 2} \epsilon^{-1})$.

Remark 12. These orbits are unstable but cf. the estimate for the characteristic multipliers, the separation is much more modest in comparison with the typical exponential separation from the trivial periodic orbit $x = 0 = y$ (see also the appendix in [26]). This result gives 1° in our main result. \diamond

PROOF. *Case (i)*

I first consider case (i). Given that w_0 is already determined by (116), I am to solve the equation (117) for \hat{z}_0 . The dependency on \hat{z}_0 enters e.g. through $\hat{\rho}_0$ in (121) with $\lambda_l = \lambda_l(\hat{z}_0)$ as in (93). I need to exclude those $\lambda_l \notin (-c^{-1}, c^{-1}) \bmod \pi$ where P_{cr} is not defined, and those $\lambda_l \notin (\pi/2 - c^{-1}, \pi/2 + c^{-1})$ where $\text{tr } \partial P$ is near ± 2 . Following

$$\partial_{\hat{z}_0} \lambda_l = \ln \epsilon^{-1} + \mathcal{O}(1), \quad (122)$$

I realise that this gives rise to the exclusion of intervals within the interval $[c_1, c_2]$ of \hat{z}_0 -values which have widths of order $c^{-1} \ln^{-1} \epsilon^{-1}$. The complement contains closed intervals with widths of order $\ln^{-1} \epsilon^{-1}$. There is an order of $\ln \epsilon^{-1}$ of such strips given an order 1 measure set of \hat{z}_0 -values. Now within each such strip for which $\lambda_l \notin (\pi/2 - c^{-1}, \pi/2 + c^{-1})$ I have

$$(F_i^{(2)})'(\hat{z}_0) = -\frac{1}{\pi} \cot(\lambda_l) \ln^2(\epsilon^{-1}) + \mathcal{O}(\ln \epsilon^{-1}). \quad (123)$$

Given that $\cot \lambda_l \neq 0$ it therefore follows that solutions of $F_i^{(2)}(\hat{z}_0) = 0 \bmod \pi$, see (117), exist within each strip. These solutions are separated by a distance of order $\ln^{-2} \epsilon^{-1}$. Furthermore, given that the order of the width of each strip is $\ln^{-1} \epsilon^{-1}$ there is an order of $\ln \epsilon^{-1}$ -many solutions within each strip. In total there are $\mathcal{O}(\ln^2 \epsilon^{-1})$ -many solutions as claimed.

These solutions can all be continued into true solutions by applying the contraction mapping theorem taking ϵ sufficiently small.

Case (ii)

In this case I have to solve two equations: (119) and (120) for \hat{z}_0 and w_0 . Cf. (61) I can, however, solve for \hat{z}_0 and $\lambda = \lambda_l$ instead. Firstly

$$(F_{ii}^{(1)})'(\hat{z}_0) = 2 \ln(e_4 \epsilon^{-1}) + \mathcal{O}(1),$$

and so there exists an order $\ln \epsilon^{-1}$ many solutions \hat{z}_0 to (119). These are separated by a distance of order $\ln^{-1} \epsilon^{-1}$ within the order 1 set of \hat{z}_0 -values. For each solution to (119) I will solve (120) with respect to λ . Again I compute the derivative

$$\partial_\lambda F_{ii}^{(2)} = -\frac{1}{\pi} \cot(\lambda) \ln \epsilon^{-1} + \mathcal{O}(1),$$

using (121). Upon excluding a region around $\lambda = \pi/2$ and $3\pi/2$ where $\cot \lambda = 0$, the same reasoning, as used above, can be applied to conclude the existence of $\mathcal{O}(\ln \epsilon^{-1})$ -many solutions for each solution \hat{z}_0 of (119). In total there is therefore an order of $\ln^2 \epsilon^{-1}$ -many solutions.

Again, the solutions can be continued into true solutions by applying the contraction mapping theorem for ϵ sufficiently small.

The solutions are unstable cf. Lemma 13. From this also follows the estimates of the characteristic multipliers. \square

From now I will focus on stable solutions.

6.3. Stable solutions

I focus on $\lambda = \lambda_l(\hat{z}_0)$ near $\pi/2$ and case (i) where $\lambda_r = \lambda_l \bmod \pi$ and $w_0 = 0, \pi/2 \bmod \pi$. Here λ_l is again given by (93) with $w_0 = 0, \pi/2 \bmod \pi$. Case (ii) can be handled in a similar way. Following (112) I wish to consider λ of the form

$$\lambda = \frac{\pi}{2} + \hat{\lambda} \ln^{-1}(\epsilon^{-1}). \quad (124)$$

Lemma 15. *Let c be large and consider $\hat{z}_0 > c^{-1}$ and $\hat{z}_0 \notin (\ln 2/(2\pi) - c^{-1}, \ln 2/(2\pi) + c^{-1})$. Then for ϵ sufficiently small, all period orbits obtained from case (i) with $\lambda = \lambda_l(\hat{z}_0)$, with λ as in (124) and where $\hat{\lambda}$ satisfies*

$$\hat{\lambda} \in \begin{cases} [-2\pi(2A + D_1) + c^{-1}, -c^{-1}] & \text{for } 2A + D_1 > 0 \\ [c^{-1}, -2\pi(2A + D_1) - c^{-1}] & \text{for } 2A + D_1 < 0 \end{cases}, \quad (125)$$

are stable. Here

$$D_1(\hat{z}_0) \equiv D(\hat{z}_0, \pi/2),$$

and

$$(2A + D_1) = \frac{3e^{4\pi\hat{z}_0} - 6e^{2\pi\hat{z}_0} + 2}{(e^{2\pi\hat{z}_0} - 1)(e^{2\pi\hat{z}_0} - 2)} \\ = 3 + 3e^{-2\pi\hat{z}_0} + \mathcal{O}(e^{-4\pi\hat{z}_0}), \quad (126)$$

$$(127)$$

for \hat{z}_0 large, where A and D are given in (107) and (110), respectively.

Remark 13. It is important to note, when comparing with the equations in [25, 26], the difference that our equations depend on the pseudo-angle λ (denoted by $\pi\eta$ in [25, 26]) strongly due to the factor $\ln \epsilon^{-1}$ of $\ln |\sin \lambda|$. Moreover, their γ_1 and γ_2 are both $\mathcal{O}(\ln \epsilon^{-1})$ in our case. Increasing γ_1 and γ_2 has the consequence of diminishing the stability region cf. Eq. (41) in [26] and in agreement with Lemma 15. \diamond

PROOF. Eq. (124) is inserted into (108) which is then used in (104). This gives

$$\text{tr } \partial P = 2 + \frac{2\hat{\lambda}}{\pi} \left(\frac{\hat{\lambda}}{\pi} + 2A + D_1 \right) + \mathcal{O}(\ln^{-1} \epsilon^{-1}).$$

A sufficient condition for stability is that $|\text{tr } \partial P| < 2$. Solving this inequality for $\hat{\lambda}$ one can then verify the statement about the stability for c large. \square

I will in the following prove 2° and 3° of the main theorem and therefore focus, as in 1° , on $\hat{z}_0 \in [c_1^{-1}, c_2]$ with $\ln 2/(2\pi) \notin [c_1^{-1}, c_2]$.

To solve for stable orbits I proceed as above but this time I start by solving $\lambda_l(\hat{z}_0) = \lambda$ for \hat{z}_0 where the right hand side λ is as in (124). Following (122) this gives $\mathcal{O}(\ln \epsilon^{-1})$ -many intervals of lengths at least $\pi|2A + D_1| \ln^{-2} \epsilon^{-1}$. These intervals are separated by intervals of lengths $\mathcal{O}(\ln^{-1} \epsilon^{-1})$ where $|\text{tr } \partial P| > 2$. Also since

$$(F_i^{(2)})'(\hat{z}_0) = -\ln(\epsilon^{-1})(2A + D_1) + \mathcal{O}(1),$$

when $\lambda_l(\hat{z}_0)$ is as in (124), these intervals will under $F_i^{(2)}$ be mapped into $\mathcal{O}(\ln \epsilon^{-1})$ -many intervals in $\mathbb{R}/(\pi\mathbb{Z})$ of lengths $\pi|2A + D_1|^2 \ln^{-1} \epsilon^{-1}$. These intervals are separated by lengths

$$\pi(2A + D_1) + \mathcal{O}(\ln^{-1} \epsilon^{-1}). \quad (128)$$

Stable solutions are then the consequence of the intersection of these mapped intervals with 0 cf. (117). The stable solutions are therefore rare compared to the unstable ones: There can be at most $\ln \epsilon^{-1}$, in contrast to $\ln^2 \epsilon^{-1}$, but this is clearly very optimistic. Before supporting this claim by further analysis I first present some numerics for (1).

6.4. Numerics for Eq. (1)

Fig. 3 shows the number of stable periodic solutions for (1) within $\hat{z}_0 \in [0.12, 2]$ and 1000 different values of $\ln^{-1} \epsilon^{-1}$. The distribution of solutions are shown in Fig. 4. The solutions are obtained from the truncations of (117) and (125). To continue them into true solutions one needs to invoke the contraction mapping theorem. For some of the extremely small values of ϵ considered it is difficult if *not* impossible to integrate the equations directly. Out of the 1000 different values of $\ln^{-1} \epsilon^{-1}$ the case with no stable solutions occurred 368 times. In 632 of the cases I found at least one stable solution.

Based on these observations, one could be led to the following conjecture:

Conjecture 1. *There exists an ϵ_0 and a number N so that the following holds true: For almost all $\epsilon \leq \epsilon_0$ the number of stable solutions is less than N .*

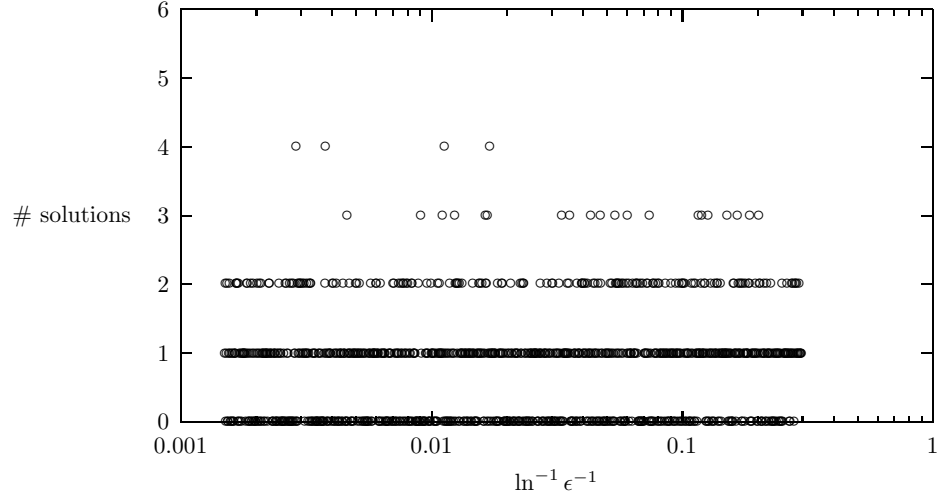


Figure 3: The number of stable solutions for $\hat{z}_0 \in [0.1, 2]$ for 1000 different values of $\ln^{-1} \epsilon^{-1}$ are shown with points.

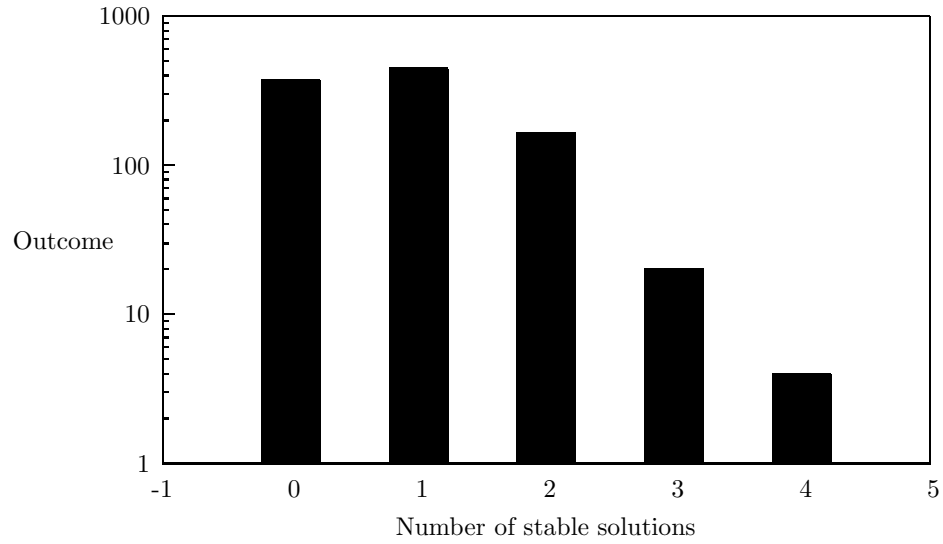


Figure 4: The number of outcomes for the different numbers of stable solutions. A total of 1000 different values of $\ln^{-1} \epsilon^{-1}$ were considered.

ϵ	POS	SPOS	SPOS $x \in (0, 2\epsilon^{1/2}]$	UPOS $x \in (0, 2\epsilon^{1/2}]$
0.08	33	0	0	33
0.04	69	2	1	46
0.02	154	5	0	64
0.01	298	8	0	72
0.005	583	18	0	84
0.0025	1118	35	0	108

Table 1: Distribution of \mathcal{T}_π -symmetric, $2\pi\epsilon^{-1}$ -periodic orbits for (1).

In Table 1 I have documented the result from computing \mathcal{T}_π -symmetric, $2\pi\epsilon^{-1}$ -periodic solutions of (1). As in [26] I have used step-size control in the initial conditions $(x, 0)$ to carefully scan for periodic orbits in the interval $x \in (0, 0.5]$. I have used a 2-stage fully implicit Gauss-Legendre symplectic method for the time integration. POS in the second column of Table 1 is the number of periodic orbits. SPOS in the third column is for the number of periodic orbits. The fourth column gives the number of stable periodic orbits for $x \in (0, 2\epsilon^{1/2}]$. The final column gives the number of unstable periodic orbits for $x \in (0, 2\epsilon^{1/2}]$. The upper value $x = 2\epsilon^{1/2}$ corresponds cf. Remark 5 to the upper bound $\hat{z}_0 = 2$ considered above.

In agreement with the analysis and the computations above the stable periodic orbits are truly rare within this interval. Also, in agreement with the results from [16, 25] the total number of periodic orbits, the number of stable period orbits, and the unstable ones all behave like ϵ^{-1} : The second and third column almost doubles when ϵ is halved. Finally, in agreement with the results of this paper the number of unstable periodic orbits within $x \in (0, 2\epsilon^{1/2}]$, the last column in Table 1, behaves like $\ln^2 \epsilon^{-1}$. A comparison is shown in Table 2.

6.5. Part 2° of the main result

To prove part 2° of the main result, suppose that an interval in $\mathbb{R}/(\pi\mathbb{Z})$, arising as the image under $F_i^{(2)}$ of a \hat{z}_0 -interval with $\lambda_l(\hat{z}_0)$ as in (124), intersects with 0. Then a stable solution exists cf. Proposition 4 and Lemma 15. Let \hat{z}_0^0 denote the left end-point of the \hat{z}_0 -interval and denote by f_0 the image of \hat{z}_0 so that $f_0 = F_i^{(2)}(\hat{z}_0^0)$. The f_0 is an end-point of the mapped interval $\mathbb{R}/(\pi\mathbb{Z})$ since $(F_i^{(2)})'(\hat{z}_0) \neq 0$. The end-points of the following \hat{z}_0 -intervals are denoted by \hat{z}_0^n and similarly f_n will denote the image of \hat{z}_0^n under $F_i^{(2)}$. Here

ϵ	UPOS $x \in (0, 2\epsilon^{1/2}]$	UPOS-fit: $\lfloor 2.39 \ln^2 \epsilon^{-1} \rfloor + 21$	Relative Error
0.08	33	36	9.1%
0.04	46	45	2.2%
0.02	64	57	11%
0.01	72	71	1.4%
0.005	84	88	4.8%
0.0025	108	106	1.9%

Table 2: Comparison of the number of unstable periodic orbits in $x \in (0, 2\epsilon^{1/2}]$ with the number predicted by the theory (third column via linear fit).

$n = 1, \dots, N$ with $N = \mathcal{O}(\ln \epsilon^{-1})$. This induces a mapping of the following form

$$\begin{aligned} f_{n+1} &= f_n + \varpi(\hat{z}_0^n) \mod \pi, \\ \hat{z}_0^{n+1} &= \hat{z}_0^n + \pi \ln^{-1} \epsilon^{-1} + \mathcal{O}(\ln^{-2} \epsilon^{-1}). \end{aligned} \quad (129)$$

In the following I will obtain an upper bound for the number steps required to obtain another solution. To obtain another solution, say \hat{z}_0^n , within the following N steps it is necessary for f_n to be within a distance of order $\ln^{-1} \epsilon^{-1}$ of f_0 . The answer depends on the arithmetic properties of the number $\varpi(\hat{z}_0^0)$. Consider therefore $c_1 > 0$ large and $d > 1$ and the following set of *Diophantine* numbers

$$D_{c_1, d} = \{z \in [0, 1) \mid |jz - i| \geq \frac{c_1^{-1}}{j^d}, i, j \in \mathbb{N}\}.$$

They have *almost* full measure: $1 - c_2(d)c_1^{-1}$ given that c_1 is large. Here the d dependency in $c_2(d)$ enters through a factor of $\sum_{j=1}^{\infty} j^{-d}$. This is why $d > 1$.

Proposition 6. *Suppose, with little loss of generality, that*

$$D_{c_1, d} \ni \pi^{-1} \varpi(\hat{z}_0^0) \mod 1,$$

with c_1 large and $d > 1$. Then

$$|f_n - f_0| \gg \ln^{-1} \epsilon^{-1},$$

for all $n \leq \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$ with c_3 large.

PROOF. I linearize (129) about \hat{z}_0^0 and use the Diophantine property to obtain the following

$$|f_n - f_0| = |n\varpi(\hat{z}_0^0) + \mathcal{O}(n^2 \ln^{-1} \epsilon^{-1})| \geq \frac{c_1^{-1}}{\pi m^d} + \mathcal{O}(m^2 \ln^{-1} \epsilon^{-1}).$$

for all $n \leq m \ll \mathcal{O}(\ln \epsilon^{-1})$. Setting $m = \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$ then implies that $m^{-d} \gg m^2 \ln^{-1} \epsilon^{-1}$ provided c_3 is large enough. From here also follows that

$$|f_n - f_0| \geq \frac{c_1^{-1}}{2\pi m^d} \gg \ln^{-1} \epsilon^{-1},$$

for all such $n \leq m = \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$. This completes the proof. \square

The values \hat{z}_0^n are separated by the distance $\pi \ln^{-1} \epsilon^{-1} + \mathcal{O}(\ln^{-2} \epsilon^{-1})$. Given that one *typically* (in the sense that the Diophantine numbers have almost full measure) have to wait longer than $m = \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$ steps between solutions (cf. Proposition 6), I can therefore, with little loss of generality, conclude that there can be at most

$$m^{-1} \ln \epsilon^{-1} = \mathcal{O}(\ln^{\frac{1+d}{2+d}} \epsilon^{-1}), \quad (130)$$

solutions within an order 1 interval of \hat{z}_0 -values. This gives 2° of the main result.

Remark 14. This result could perhaps be improved to something like Conjecture 1. It would require a theory for the asymptotic distribution of points on the circle $\mathbb{R}/(\pi\mathbb{Z})$ of “forced” circle maps of the form (129). \diamond

6.6. Distribution of stable solutions - Part 3° of the main result

Thus far I have kept ϵ fixed but small without being able to say anything about the existence or non-existence of stable solutions close to the bifurcating normally elliptic slow manifold. In fact, the numerics from above seem to indicate that both situations with existence and non-existence can occur. In this section, I will, however, study how stable solutions can be created when varying ϵ . This will cover part 3° of the main result. I still focus on case (i) although the result is also true for case (ii). Consider $\hat{z}_0^0 = \hat{z}_0^0(\epsilon)$ from section Section 6.5 above. It is a solution of $\lambda_l(\hat{z}_0^0) = \frac{\pi}{2} + \hat{\lambda} \ln^{-1} \epsilon^{-1}$ with $\hat{\lambda}$ held fixed

at a value in the interior of the interval in (125). This gives the following expression for $(\hat{z}_0^0)'(\epsilon) = \frac{d}{d\epsilon} \hat{z}_0^0(\epsilon)$:

$$(\hat{z}_0^0)'(\epsilon) = -(\partial_{\hat{z}_0} \lambda_l)^{-1}(\partial_\epsilon \lambda_l + \mathcal{O}(\epsilon^{-1} \ln^{-2} \epsilon^{-1})) = \ln^{-1}(\epsilon^{-1}) \epsilon^{-2} e_3 + \mathcal{O}(\ln^{-2}(\epsilon^{-1}) \epsilon^{-2}).$$

The rate of change of $f_0 = f_0(\epsilon)$, which is the image of $\hat{z}_0^0(\epsilon)$ under $F_i^{(2)}$, is then also easily obtained:

$$f_0'(\epsilon) = ((2A + D_1)e_3 - e_1)\epsilon^{-2} + \mathcal{O}(\ln^{-1}(\epsilon^{-1})\epsilon^{-2}). \quad (131)$$

Theorem 16. *Take any $\epsilon = \epsilon_1$ sufficiently small and set $I_1 = [\epsilon_1 - c_1 \epsilon_1^2, \epsilon_1 + c_1 \epsilon_1^2]$. Then within I_1 there will exist $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many closed intervals of lengths $\geq c_3^{-1} \epsilon_1^2 \ln^{-1} \epsilon_1$ for which there exists stable solutions. Here c_1, c_2 and c_3 may be large but they can be taken to be independent of ϵ_1 .*

PROOF. Consider $f_0 = f_0(\epsilon)$ with derivative f_0' as in (131). Recall that f_0 is an end-point of a mapped interval that has length of order $\ln^{-1} \epsilon^{-1}$. If this interval does not intersect with 0 in $\mathbb{R}/(\pi\mathbb{Z})$ then one can simply, cf. (131), alter ϵ from ϵ_1 by an amount of $c_1 \epsilon_1^2$ to “push” this towards 0 (provided that $(2A + D_1)e_3 - e_1 \neq 0$) and ensure the existence of a stable solution $(\hat{z}_0^0, \hat{\lambda})$. Since the mapped interval has a length of order $\ln^{-1} \epsilon_1$ this stable solution persists within a closed interval of ϵ -values with size of order $c_3^{-1} \epsilon_1^2 \ln^{-1} \epsilon_1^{-1}$. The \hat{z}_0 -interval based at \hat{z}_0^0 was arbitrary and the argument applies to all of the $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many intervals with end-points at \hat{z}_0^n . The solutions can be continued into true solutions by applying the contraction mapping theorem. \square

By a similar argument to the one used in Proposition 6 it will also follow that if one takes $\lfloor c_4^{-1} \ln^{(2+d)^{-1}} \epsilon_1^{-1} \rfloor$ -following f_n 's, then they will typically be distant from each-other by a length of $\gg \ln^{-1} \epsilon_1^{-1}$. Here $d > 1$. The relative measure within I_1 of the union of the $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many closed intervals of stable orbits will therefore typically be larger than $c_5^{-1} \ln^{-\frac{1+d}{2+d}} \epsilon_1^{-1} \gg \ln^{-1} \epsilon^{-1}$.

6.7. Part 4° of the main result

The last part 4° of the main result is the consequence of the following simple observation: The separation (128) between two consecutive mapped intervals having end-points e.g. at f_n and f_{n+1} are

$$3\pi + 3\pi e^{-2\pi \hat{z}_0} + \mathcal{O}(e^{-4\pi \hat{z}_0}) = 3\pi e^{-2\pi \hat{z}_0} + \mathcal{O}(e^{-4\pi \hat{z}_0}) \pmod{\pi},$$

cf. (126), dropping for simplicity the super-script on \hat{z}_0^n . By taking \hat{z}_0 “large” of size $\mathcal{O}(\ln \ln \epsilon^{-1})$ I can therefore guarantee that the separation is small being of order $\ln^{-1} \epsilon^{-1}$. Indeed, replace \hat{z}_0 by $\frac{1}{2\pi} \ln(\hat{Z}_0^{-1} \ln \epsilon^{-1})$. Then

$$\begin{aligned} 3\pi e^{-2\pi \hat{z}_0} + \mathcal{O}(e^{-4\pi \hat{z}_0}) &= 3\pi e^{\ln(\hat{Z}_0 \ln^{-1} \epsilon^{-1})} + \mathcal{O}(e^{\ln(\hat{Z}_0^2 \ln^{-2} \epsilon^{-1})}) \\ &= 3\pi \hat{Z}_0 \ln^{-1} \epsilon^{-1} + \mathcal{O}(\ln^{-2} \epsilon^{-1}). \end{aligned}$$

Hence for \hat{Z}_0 large two consecutive intervals, with left end-points at f_n and f_{n+1} , are guaranteed to overlap.

Remark 15. Taking \hat{z}_0 of this form implies, cf. e.g. (31), that the motion of (x, y) undergo fast oscillations near the passage through $u = 0$. Also cf. e.g. Remark 5 the distance to the slow manifold is $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$. \diamond

All of the estimates above can be modified to account for \hat{z}_0 of this form. For P_i and P_o this follow immediately from the analysis above. For P_{cr} one needs to analyze the correction term. This leads into similar calculations as the ones performed for P_i and P_o . The changes are therefore minor and I therefore leave the details out of the manuscript.

Now, remember that within an $\mathcal{O}(1)$ -interval of initial \hat{z}_0 -values there are $\mathcal{O}(\ln \epsilon^{-1})$ -many mapped intervals of lengths $\approx 12\pi \ln^{-1} \epsilon^{-1}$ cf. (125) and (126). Therefore by taking \hat{Z}_0 sufficiently small, but independent of ϵ , and a sufficiently large interval of initial \hat{z}_0 -values, the union of these many intervals can be guaranteed to cover the whole circle $\mathbb{R}/(\pi\mathbb{Z})$. In particular, there is at least one interval which intersects 0. This provides the existence of a stable solution and proves 4°.

I have collected the conclusions of 2° and 4° on case (i) in an illustration shown in Fig. 5.

7. Stability islands

Generically stability islands surround stable fix points of P or P^2 . How large are these stability islands? To address this I first need to introduce a further blowup or scaling:

$$\hat{z}_0 = \ln^{-2}(\epsilon^{-1}) \hat{Z}, \quad w_0 = \ln^{-1}(\epsilon^{-1}) \hat{W}. \quad (132)$$

The purpose of this is to obtain an order 1 blowup Poincaré mapping denoted by \hat{P} when λ_l is given as in (124). Indeed, from Lemma 12 it follows that

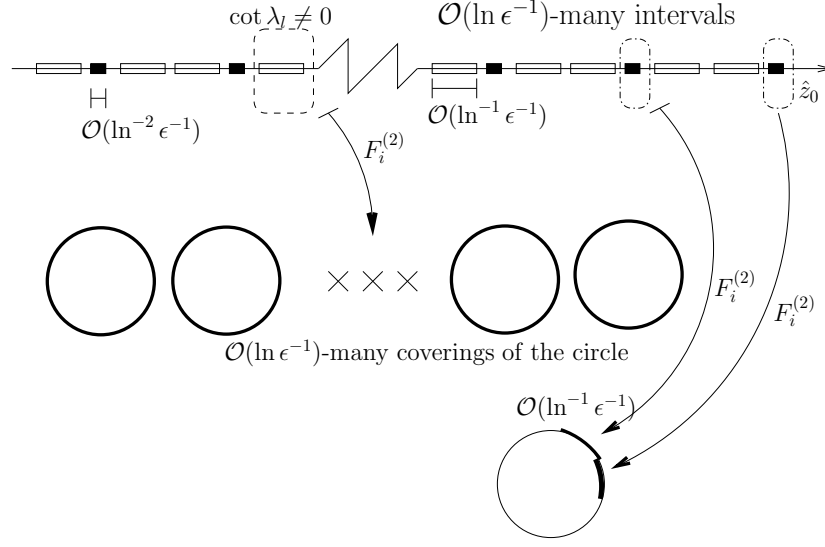


Figure 5: Unstable and periodic orbits obtained from case (i). On the \hat{z}_0 -axis on the top the non-filled rectangles indicate the \hat{z}_0 -values where $\cot^2 \lambda_l \geq c^{-1}$, giving rise to unstable periodic orbits, while the black, filled rectangles correspond to regions where stability could be attained according to (125). The image under $F_i^{(2)}$ of one of the non-filled rectangles covers the circle $\mathbb{R}/(\pi\mathbb{Z})$ an order of $\mathcal{O}(\ln \epsilon^{-1})$ -many times cf. (123). This gives rise to a total of $\mathcal{O}(\ln^2 \epsilon^{-1})$ -many unstable solutions. The image under $F_i^{(2)}$ of one of the smaller filled rectangles, however, only covers a small $\mathcal{O}(\ln^{-1} \epsilon^{-1})$ -portion of the circle. According to 4°, however, taking large \hat{z}_0 -values of order $\ln \ln \epsilon^{-1}$, two upon each-other following images under $F_i^{(2)}$ can be guaranteed to overlap in the way shown by the thick tubes of $F_i^{(2)}$ -images of the two rectangles highlighted by the dash-dotted lines in the top right corner. This way by taking $\mathcal{O}(\ln \epsilon^{-1})$ -many mapped intervals the circle can be covered completely by $F_i^{(2)}$ -images of black rectangles and the existence of at least one stable solution follows.

the Jacobian of $\hat{P} = \hat{P}(\hat{Z}, \hat{W})$ with respect to these variables satisfy:

$$\begin{aligned} \partial_{(\hat{Z}, \hat{W})} \hat{P} = & \begin{pmatrix} AD_1 + \frac{(2A+D_1)\hat{\lambda}}{\pi} + \frac{\hat{\lambda}^2}{\pi^2} & -\frac{D_1\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} \\ -2A(2A+D_1) - \frac{(4A+D_1)\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} & 2 - AD_1 + \frac{(2A+D_1)\hat{\lambda}}{\pi} + \frac{\hat{\lambda}^2}{\pi^2} \end{pmatrix} \\ & + \mathcal{O}(\ln^{-1} \epsilon^{-1}), \end{aligned}$$

in case (i) and

$$\partial_{(\hat{Z}, \hat{W})} \hat{P} = \begin{pmatrix} AD_1 - 2\frac{A\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} & \frac{\hat{\lambda}^2}{\pi^2} \\ -2a(2A + D_1) + \frac{\hat{\lambda}^2}{\pi^2} & 2 - AD_1 + 2\frac{A\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} \end{pmatrix} + \mathcal{O}(\ln^{-1} \epsilon^{-1}),$$

in case (ii). Also

$$\hat{\lambda} = \pi/2 \ln \epsilon^{-1} - \hat{W} + (1 + \ln^{-1}(\epsilon^{-1}) \ln e_4) \hat{Z} + \ln(\epsilon^{-1}) G(\ln^{-2}(\epsilon^{-1}) \hat{Z}).$$

Note in particular how the trace in case (i) here agrees with the one presented in Lemma 15. Consider $(\hat{Z}, \hat{W}) = (\hat{Z}_e, \hat{W}_e)$ a stable fix points of \hat{P} or \hat{P}^2 . Then generically KAM-theory can be applied to \hat{P} or \hat{P}^2 to conclude the existence of invariant curves surrounding $(\hat{Z}, \hat{W}) = (\hat{Z}_e, \hat{W}_e)$. The last one of such invariant curves creates a resonance island that measures $\mathcal{O}(1)$ in the (\hat{Z}, \hat{W}) -space; $\mathcal{O}(\ln^{-3} \epsilon^{-1})$ in (\hat{z}_0, w_0) -space cf. (132); $\mathcal{O}(\epsilon \ln^{-3} \epsilon^{-1})$ in the original variables (x, y) cf. (33).

8. Future work

The existence of the stable orbits in 4° provides a beginning of a connection with the work in [16, 25, 26]. Future work should seek to describe a more detailed connection to this work, by providing a description of the distribution of periodic orbits further away from the slow manifold. It is reasonable to believe that this requires a combination of the techniques used here with those used in [16, 25, 26].

9. Acknowledgement

I would like to thank Prof. A. I. Neishtadt for pointing me in the direction of [24] and for suggestions leading to an improved manuscript.

Appendix A. Proof of Lemma 2

I focus on the estimates for the Jacobian. The first statement about the growth of \hat{x} and \hat{y} will follow from similar estimates. First I take $\hat{u} \leq -\check{u}_*\delta$ with \check{u}_* large as in Lemma 1. Then from the results presented in that lemma I obtain the following asymptotics

$$\begin{aligned} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} &= \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{x}(-\hat{u}_*) \\ \hat{y}(-\hat{u}_*) \end{pmatrix}^{-1} \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} \delta^{-1/4} & \delta^{-1/4} \\ \delta^{1/4} & \delta^{1/4} \end{pmatrix} \right), \end{aligned}$$

For $\hat{u} \in (-\check{u}_*\delta, \check{u}_*\delta)$ I use the coordinates: $(\check{x}, \check{y}) = (\delta^{1/4}\hat{x}, \delta^{-1/4}\hat{y})$ and \hat{u} replaced by $\check{u} = \delta^{-1}\hat{u}$, also used in [24], to control the assignment $(\check{x}, \check{y})(-\check{u}_*) \mapsto (\check{x}, \check{y})(\hat{u}\delta^{-1})$, with a bound that is independent of δ . Returning to my coordinates I then obtain the following

$$\partial_{(\hat{x}(-\check{u}_*\delta), \hat{y}(-\check{u}_*\delta))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} = \mathcal{O} \begin{pmatrix} 1 & \delta^{-1/2} \\ \delta^{1/2} & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} &= \partial_{(\hat{x}(-\check{u}_*\delta), \hat{y}(-\check{u}_*\delta))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(-\check{u}_*\delta) \\ \hat{y}(-\check{u}_*\delta) \end{pmatrix} \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} 1 & \delta^{-1/2} \\ \delta^{1/2} & 1 \end{pmatrix} \begin{pmatrix} \delta^{-1/4} & \delta^{-1/4} \\ \delta^{1/4} & \delta^{1/4} \end{pmatrix} \right) \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} \delta^{-1/4} & \delta^{-1/4} \\ \delta^{1/4} & \delta^{1/4} \end{pmatrix} \right). \end{aligned}$$

Finally, I consider $\hat{u} \geq \check{u}_*\delta$ and use the following

$$\begin{aligned} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} &= \partial_{(\hat{\rho}_0, \phi_0)} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{\rho}_0 \\ \phi_0 \end{pmatrix} \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{x}(-\hat{u}_*) \\ \hat{y}(-\hat{u}_*) \end{pmatrix}^{-1} \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} \hat{u}^{-1/4} \ln(\delta^{-1}\hat{u}) & \hat{u}^{-1/4} \ln(\delta^{-1}\hat{u}) \\ \hat{u}^{1/4} \ln(\delta^{-1}\hat{u}) & \hat{u}^{1/4} \ln(\delta^{-1}\hat{u}) \end{pmatrix} \right). \end{aligned}$$

Setting $\hat{u} = \hat{u}_*$ here gives (67). For (68) I combine the asymptotics to obtain the following bound

$$\left| \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \right| \leq c\delta^{-1/4} \ln \delta^{-1},$$

uniformly in \hat{u} . The estimate of the inverses can be derived in a similar manner using the fact that the Jacobian has determinant equal to 1.

Appendix B. Approximation of P_o

I start by presenting two lemmata similar to Lemma 5 and Lemma 6:

Lemma 17. *Let q and p be positive real numbers satisfying $0 < q < p$. Then there exists a constant $c = c(\hat{u}_*)$ so that*

$$\hat{F}(\hat{u})^{-2p} \leq c^{p-q} \hat{F}(\hat{u})^{-2q},$$

for all $\hat{u} \in [-\mu^{-2}(\pi - \tau/2), -\hat{u}_*]$.

Lemma 18. *Let $q \in \overline{\mathbb{R}}_+$. Given an integrable function $r = r(\hat{u})$ satisfying the following estimate*

$$|r(\hat{u})| \leq \hat{F}(\hat{u})^{-2q}, \quad \hat{u} \in [\hat{u}_*, \mu^{-2}\tau/2].$$

If $q < 1$ then there exists a $c_1 = c_1(q)$ so that

$$\left| \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} r(\hat{u}) d\hat{u} \right| \leq c_1 \mu^{-2(1-q)}.$$

If $q = 1$ then there exists a c_2 so that

$$\left| \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} r(\hat{u}) d\hat{u} \right| \leq c_2 \ln(\mu^{-2}\hat{u}_*^{-1}).$$

Finally if $q > 1$ then the corresponding integral is uniformly bounded with respect to ϵ : There exists a c_3 so that

$$\left| \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} r(\hat{u}) d\hat{u} \right| \leq c_3 (q-1)^{-1} \hat{u}_*^{1-q}.$$

The proofs of these lemmata are almost identical to the proofs of Lemma 5 and Lemma 6 and therefore left out.

I then recall the form of \hat{H} in (40):

$$\hat{H} = h_0(\hat{u}, \hat{v}, \hat{z}_0) + r_0(\hat{u}, \hat{v}, \hat{z}_0, w_0) + \mathcal{O}(\hat{F}(\hat{u})^{-5}\delta^3), \quad (\text{B.1})$$

and note that

$$\begin{aligned} \bar{r}_0 &= \frac{1}{2\pi} \int_0^{2\pi} r_0(\hat{u}, \hat{z}_0, \tau) d\tau = \frac{3}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} \hat{z}_0^2 - \frac{1}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} f(u) M_0(u) \hat{z}_0^2 \\ &= \frac{3}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z}_0^2. \end{aligned}$$

I then introduce the following generating function

$$G(\hat{u}, \hat{v}, \hat{z}_0, w_1) = \delta^{-3/2} \hat{u} \hat{v} + \hat{z}_0 w_1 + \hat{F}(\hat{u})^{-1} \int_0^{\tilde{\phi}_1} \tilde{r}_0(\hat{u}, \hat{z}_0, \tau) d\tau,$$

$$\tilde{r}_0 = r_0 - \bar{r}_0.$$

This generates a symplectic transformation $(\hat{u}, \hat{v}_0, \hat{z}_0, w_0) \mapsto (\hat{u}, \hat{v}, \hat{z}, w)$ with $\hat{z}_0 = \hat{z} + \mathcal{O}(\delta^{3/2})$ transforming H (B.1) into

$$\hat{H} = \hat{v} + \hat{F}(\hat{u}) \hat{z} + \frac{3}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z}^2 + \mathcal{O}(\hat{F}(\hat{u})^{-5} \delta^3).$$

The equations of motion are

$$\begin{aligned} \frac{d\hat{z}}{d\hat{u}} &= \mathcal{O}(\hat{F}(\hat{u})^{-5} \delta^{3/2}), \\ \frac{dw}{d\hat{u}} &= -\delta^{-3/2} \hat{F}(\hat{u}) - \frac{3}{2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z} + \mathcal{O}(\hat{F}(\hat{u})^{-5} \delta^{3/2}). \end{aligned} \quad (\text{B.2})$$

In accordance with the definition of P_o I consider \hat{u} here from $\hat{u} = -\mu^{-2}(\pi - \tau/2)$ to $\hat{u} = -\hat{u}_*$.

Lemma 19. \hat{z}_0 is conserved on the interval from $\hat{u} = -\mu^{-2}(\pi - \tau/2)$ to $\hat{u} = -\hat{u}_*$ up to an error of order $\mathcal{O}(\delta^{3/2})$.

PROOF. I can take $p = 5/2 > 1$ in Lemma 18 to control the variation of \hat{z} by an error of order $\delta^{3/2}$. Since $\hat{z}_0 = \hat{z} + \mathcal{O}(\delta^{3/2})$ the result follows. \square

By a similar argument, I estimate the effect of the remainder in (B.2) by $\mathcal{O}(\delta^{3/2})$ and I compute the variation in the angle w by

$$\begin{aligned} w(-\hat{u}_*) &= w(-\mu^{-2}(\pi - \tau/2)) - \int_{-\mu^{-2}(\pi - \tau/2)}^{-\hat{u}_*} \left(\delta^{-3/2} \hat{F}(\hat{u}) \right. \\ &\quad \left. + \frac{3}{2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z}_0 + \mathcal{O}(\hat{F}(\hat{u})^{-2} \delta^{3/2}) \right) d\hat{u} + \mathcal{O}(\delta^{3/2}). \end{aligned} \quad (\text{B.3})$$

As above in Section 5.1, the remainder $\mathcal{O}(\hat{F}(\hat{u})^{-2} \delta^{3/2})$ in the integral comes from $\hat{z}(\hat{u}) = \hat{z}_0 + \mathcal{O}(\delta^{3/2})$, with $\hat{z}_0 = \text{const.}$ on this interval. This can be estimated from above by a term of order $\delta^{3/2} \ln \mu^{-1}$ using $q = 1/2$ in Lemma 18. The following lemma gives asymptotics of the two other integrals appearing in (B.3).

Lemma 20.

$$\delta^{-3/2} \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u}) d\hat{u} = \mu^{-3} \delta^{-3/2} e_3 - \frac{2}{3} \delta^{-3/2} \hat{u}_*^{3/2} + \mathcal{O}(\epsilon^{2/3} \delta^{-5/2}).$$

with

$$e_3 = \int_0^{\pi-\tau/2} (-f(-u))^{1/2} du.$$

Moreover, there exists some positive constant e_4 such that

$$\int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u})^{-2} d\hat{u} = \ln(e_4 \mu^{-2} \delta^{-1}) - \ln(\delta^{-1} \hat{u}_*) + \mathcal{O}(\mu^2).$$

PROOF. I use (34):

$$\begin{aligned} \delta^{-3/2} \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u}) d\hat{u} &= \mu^{-3} \delta^{-3/2} \int_{u_*}^{\pi-\tau/2} (-f(-u))^{1/2} du \\ &= \mu^{-3} \delta^{-3/2} e_3 - \mu^{-3} \delta^{-3/2} \int_0^1 (-f(-u_* s))^{1/2} ds u_*, \end{aligned} \tag{B.4}$$

here $u_* = \mu^2 \hat{u}_*$. Since $f(u) = u + \mathcal{O}(u^2)$ cf. (11) for small u , I can for the last integral in (B.4) use the same argument used for (83) to complete the first part of the proof.

For the second part:

$$\int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) d\hat{u} = \int_{u_*}^{\pi-\tau/2} (-f(-u))^{-1} (1 + \mathcal{O}(u)) du.$$

I write $(-f(-u))^{-1} = u^{-1} + \mathcal{O}(1)$ for u small and complete the result as in the proof of Lemma 10. \square

Following this lemma I can therefore write (B.3) as

$$\begin{aligned} w_0(-\hat{u}_*) &= w_0(-\mu^{-2}(\pi - \tau/2)) - \epsilon^{-1} e_3 - \frac{3}{2} \ln(e_4 \mu^{-2} \delta^{-1}) \hat{z}_0 + \frac{2}{3} \delta^{-3/2} \hat{u}_*^{3/2} \\ &+ \frac{3}{2} \ln(\delta^{-1} \hat{u}_*) \hat{z}_0 + \mathcal{O}(\delta^{3/2} \ln \mu^{-1}). \end{aligned} \tag{B.5}$$

using here that $w_0 = w + \mathcal{O}(\delta^{3/2})$. Colleting the results, I obtain Proposition 3.

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